

武汉大学

信号处理研究室

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《数字信号处理》

Digital Signal Processing

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本科生课程, 2018 秋



Time to Zzz...

TIME!



teaching the 'Z-TRANSFORM'...

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Outline

1 z-Transform

- Definition
- Property of z-Transform
- Rational z-Transform
- Inverse z-Transform

2 LTI System in Transfer Domain

- Transfer Function
- Analysis of LTI

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1 z-Transform

■ Definition

- Property of z-Transform
- Rational z-Transform
- Inverse z-Transform

2 LTI System in Transfer Domain

- Transfer Function
- Analysis of LTI

DTFT 变换

DTFT 变换： 序列 $x[n]$ 的 **DTFT** 变换

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

存在条件：

① 绝对可和条件：

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty, \text{ (一致收敛)}$$

② 平方可和条件：

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty, \text{ (均方收敛)}$$

例子：

$$x[n] = \alpha^n u[n], \quad |\alpha| > 1$$

Z 变换

Z 变换：序列 $x[n]$ 的 **Z 变换** ($z \in \mathbb{C}$)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

与 DTFT 的关系：将变量 z 记作 $z = r \cdot e^{j\omega}$ ，那么 Z 变换可以写成如下形式

$$X(r \cdot e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} \longleftrightarrow DTFT\{x[n]r^{-n}\}$$

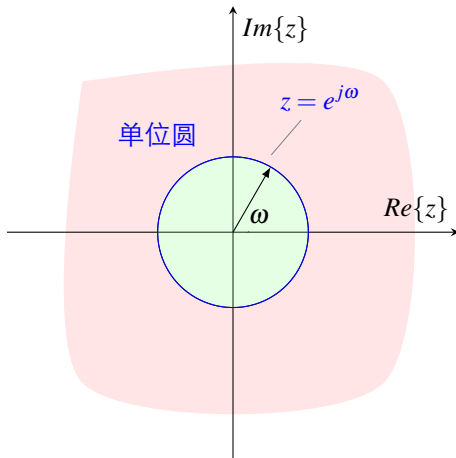
NOTES

Z 变换为 DTFT 变换的扩展。

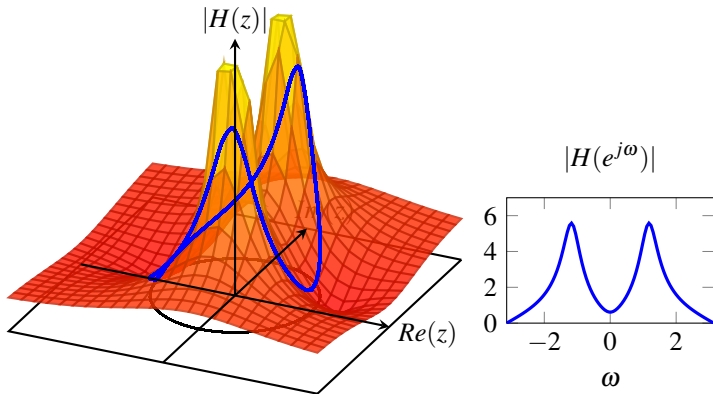
- Z 变换实际上是对序列 $x[n]r^{-n}$ 的 DTFT 变换；
- 当 $r = 1$ 时，Z 变换与 DTFT 变换等效。

Z 变换与 DTFT 变换

z -Domain



Z 变换与 DTFT 变换



Z 变换的收敛域

- 序列 $x[n]$ 的 Z 变换为

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} \end{aligned}$$

- Z 变换存在的条件:** 序列 $x[n]r^{-n}$ **绝对可和**

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

那么, 序列 $x[n]$ 的 Z 变换存在。

- Z 变换的收敛域:** 使得序列 $x[n]$ 的 Z 变换存在的 z 的所有值的集合, 称为 $x[n]$ 的 Z 变换的收敛域。

Z 变换与 DTFT 变换

$x[n]$ 绝对可和 $\stackrel{?}{\Leftrightarrow} x[n]r^{-n}$ 绝对可和

① 序列的 Z 变换存在 \Rightarrow DTFT 变换一定存在?

② 序列的 DTFT 变换存在 \Rightarrow Z 变换一定存在?

Z 变换与 DTFT 变换

$x[n]$ 绝对可和 $x[n]r^{-n}$ 绝对可和

- ① 序列的 Z 变换存在 \Rightarrow DTFT 变换一定存在? **NO**

$$x[n] = \alpha^n u[n], \quad \alpha > 1$$

- ② 序列的 DTFT 变换存在 \Rightarrow Z 变换一定存在? **NO**

$$x[n] = \frac{\sin \omega n}{\pi n}$$

Z Transform: examples

- **Example:** $x[n] = \delta[n], X(z) = 1$ and $\text{ROC} = \mathbb{C} = \text{entire } z\text{-plane}$.
- **Example:** $x[n] = \delta[n - k], X(z) = z^{-k}$ and

$$\text{ROC} = \begin{cases} \mathbb{C}, & k = 0 \\ \mathbb{C} - \{0\}, & k > 0 \\ \mathbb{C} - \{\infty\}, & k < 0 \end{cases}$$

- **Example:** $x[n] = \{4, \underline{3}, 0, \pi\}$. $X(z) = 4z + 3 + \pi z^{-2}$,
 $\text{ROC} = \mathbb{C} - \{0\} - \{\infty\}$.
- **NOTE:** 有限长序列 Z 变换的 ROC 为整个 z-平面，可能会除掉 $z = 0$ 和 $z = \infty$ 。

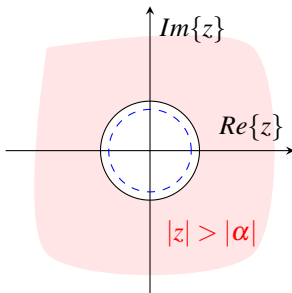
Z Transform: an example

右边序列（因果） $x[n] = \alpha^n \mu[n]$ 的 Z 变换：

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha z^{-1}| < 1$$

对应的收敛域 ROC 为： $|z| > |\alpha|$

ROC of $X(z)$

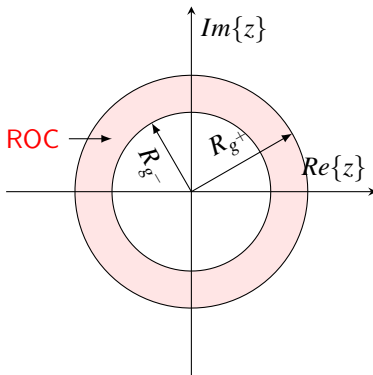


ROC 的性质

ROC 性质 1

Z-变换的收敛域 (ROC) 通常是一个环状的区域,

$$0 \leq R_{g^-} \leq r \leq R_{g^+} < \infty$$



Why an Annulus?

- Let $z = re^{j\theta}$

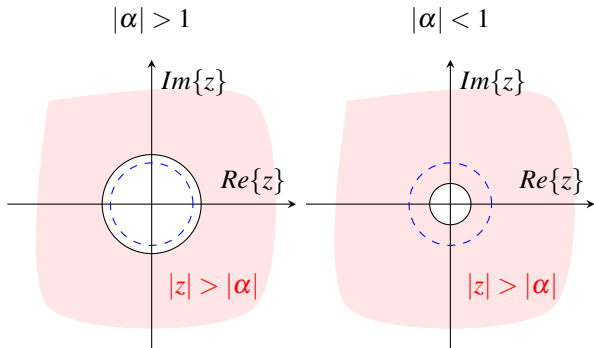
$$\begin{aligned}|X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]|r^{-n} \\&= \sum_{n=-\infty}^{-1} |x[n]|r^{-n} + \sum_{n=0}^{\infty} |x[n]|r^{-n} \\&= \sum_{n=1}^{\infty} |x[-n]|r^n + \sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}\end{aligned}$$

- Why an annulus?** Because the value of $|X(z)|$ is independent to θ .

ROC 的性质

ROC 的性质 2

当且仅当序列 $x[n]$ 的 Z 变换收敛域包含单位圆的时候，序列 $x[n]$ 的傅里叶变换是绝对收敛的。



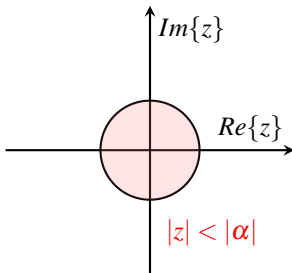
例子 2

左边序列（非因果） $y[n] = -\alpha^n \mu[-n-1]$ 的 Z 变换：

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} -\alpha^n \mu[-n-1] z^{-n} = - \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = - \sum_{m=1}^{\infty} \alpha^{-m} z^m \\ &= - \frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha^{-1} z| < 1 \end{aligned}$$

对应的收敛域 ROC 为： $|z| < |\alpha|$

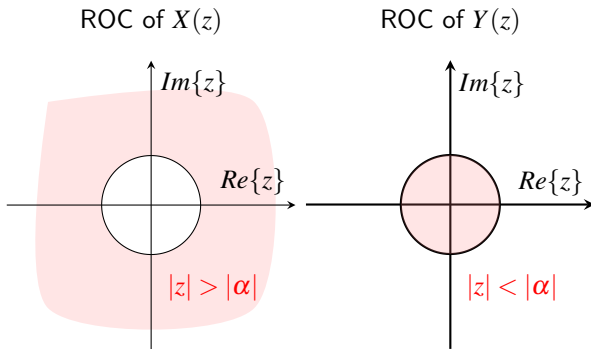
ROC of $Y(z)$



ROC 的性质

ROC 的性质 3

因果序列 $x[n]$ 的收敛域为以原点为中心过距离中心最远的极点的圆以外的所有区域；非因果序列 $x[n]$ 的收敛域为以原点为中心过距离中心最近的极点的圆以内的所有区域。



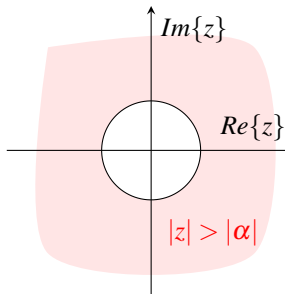
收敛域 ROC 的重要性

两个不同的序列，其 Z 变换的形式完全一样：

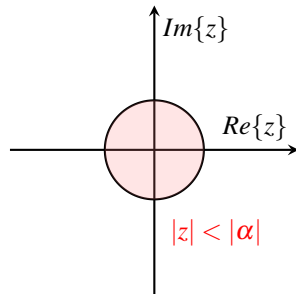
$$x[n] = \alpha^n \mu[n] \longleftrightarrow X(z) = \frac{1}{1 - \alpha z^{-1}},$$

$$y[n] = -\alpha^n \mu[-n-1] \longleftrightarrow Y(z) = \frac{1}{1 - \alpha z^{-1}}$$

ROC of $X(z)$



ROC of $Y(z)$

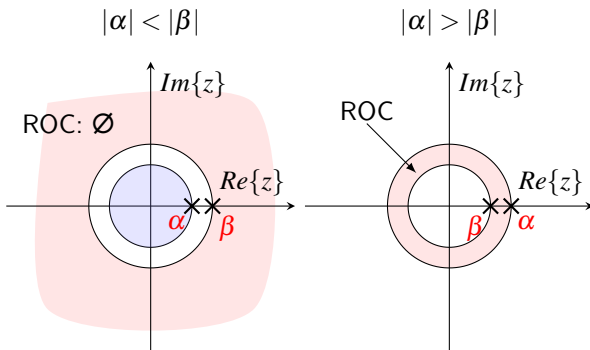


例子 3

双边序列 $h[n] = x[n] + y[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1]$ 的 Z 变换:

$$H(z) = X(z) + Y(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{1}{1 - \beta z^{-1}} = \frac{2 - (\alpha + \beta)z^{-1}}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}$$

序列 $h[n]$ 的 Z 变换 ROC = $\text{ROC}_x \cap \text{ROC}_y$ 。



ROC 的性质

ROC 的性质 4

双边序列 $x[n]$ 的 Z 变换的收敛域有两种情况：

- ① 环形区域；
- ② 空集合，即 Z 变换不存在。

ROC 的性质 5

有限长序列 $x[n]$ 的 Z 变换的收敛域为整个复平面。

$$x[n], N_1 \leq n \leq N_2 \longleftrightarrow X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} < \infty$$

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Linearity

- Linearity:

$$g[n] \longleftrightarrow G(z), \quad \text{ROC: } \mathcal{R}_g$$

$$h[n] \longleftrightarrow H(z), \quad \text{ROC: } \mathcal{R}_h$$

$$\alpha g[n] + \beta h[n] \longleftrightarrow \alpha G(z) + \beta H(z), \quad \text{ROC: } \mathcal{R}_g \cap \mathcal{R}_h$$

- Example: causal sinusoid

$$x[n] = \cos(\omega_0 n + \phi)u[n]$$

- Formula $X(z)$: according to Euler's identity

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}e^{j\phi}}{1 - e^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}e^{-j\phi}}{1 - e^{-j\omega_0}z^{-1}} \\ &= \frac{\cos \phi - z^{-1} \cos(\omega_0 - \phi)}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \end{aligned}$$

- ROC: $|z| < 1$

Time Shifting

- Time-shifting:

$$g[n - n_0] \longleftrightarrow z^{-n_0} G(z), \quad \text{ROC: } \mathcal{R}_g \text{ except possibly } z = 0 \text{ or } \infty$$

- Proof:

$$\begin{aligned} g[n - n_0] &\longleftrightarrow \sum_{n=-\infty}^{\infty} g[n - n_0] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} g[n - n_0] z^{-(n-n_0)} z^{-n_0} = z^{-n_0} G(z) \end{aligned}$$

- NOTE:** That's why z^{-1} represent 1 sample delay.

Time Reversal

- Time reversal:

$$g[-n] \longleftrightarrow G(1/z), \quad \text{ROC: } 1/\mathcal{R}_g$$

- Proof:

$$\begin{aligned} g[-n] &\longleftrightarrow \sum_{n=-\infty}^{\infty} g[-n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} g[-n](z^{-1})^{-(-n)} = G(1/z) \end{aligned}$$

- ROC:** if ROC of $G(z)$ is $\{r_1 < |z| < r_2\}$, then the ROC of $G(1/z)$ is $\{1/r_1 < |z| < 1/r_2\}$

Conjugation

- Conjugation:

$$g^*[n] \longleftrightarrow G^*(z^*), \quad \text{ROC: } \mathcal{R}_g$$

- Proof:

$$\begin{aligned} g^*[n] &\longleftrightarrow \sum_{n=-\infty}^{\infty} g^*[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} g^*[n]((z^*)^{-n})^* \\ &= \left(\sum_{n=-\infty}^{\infty} g[n](z^*)^{-n} \right)^* \\ &= G^*(z^*) \end{aligned}$$

- ROC: do not change!!!

Differentiation in z -domain

- Conjugation:

$$ng[n] \longleftrightarrow -z \frac{dG(z)}{dz}, \quad \text{ROC: } \mathcal{R}_g \text{ except possibly } z = 0 \text{ or } \infty$$

- Proof:

$$\begin{aligned} \frac{dG(z)}{dz} &= \frac{d \sum_{n=-\infty}^{\infty} g[n] z^{-n}}{dz} \\ &= \sum_{n=-\infty}^{\infty} g[n] \frac{d(z^{-n})}{dz} \\ &= \sum_{n=-\infty}^{\infty} g[n] (-nz^{-n-1}) \\ &= -z^{-1} \sum_{n=-\infty}^{\infty} ng[n] z^{-n} \end{aligned}$$

- ROC: do not change!!!

Differentiation in z -domain: example

- Unit ramp signal

$$x[n] = nu[n]$$

- Since the transform of $u[n]$

$$U(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC: } |z| > 1$$

- Using differentiation property

$$X(z) = -z \frac{d}{dz} U(z) = -z \frac{-z^{-2}}{(1-z^{-1})^2} = \frac{z^{-1}}{(1-z^{-1})^2}$$

- ROC: $|z| > 1$

Convolution

- Convolution:

$$g[n] \otimes h[n] \longleftrightarrow G(z)H(z)$$

- Proof:

$$\begin{aligned} g[n] \otimes h[n] &\longleftrightarrow \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} g[k]h[n-k] \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} g[k] \left(\sum_{n=-\infty}^{\infty} h[n-k]z^{-n} \right) \\ &= \sum_{k=-\infty}^{\infty} g[k]z^{-k}H(z) \\ &= G(z)H(z) \end{aligned}$$

- ROC:** contains at least as much of the z-plane as the intersection $\mathcal{R}_g \cap \mathcal{R}_h$

Convolution: example

- Consider

$$x[n] = u[n] * u[n-1] = nu[n]$$

- Z transform of $g[n] = u[n-1]$ (time shift)

$$G(z) = z^{-1}U(z) = \frac{z^{-1}}{1-z^{-1}}$$

- Z transform of $x[n]$ (convolution)

$$X(z) = \frac{1}{1-z^{-1}} \cdot \frac{z^{-1}}{1-z^{-1}} = \frac{z^{-1}}{(1-z^{-1})^2}$$

- ROC:** ROC for both $u[n]$ and $u[n-1]$ is $\{|z| > 1\}$.

Convolution: example

- Consider input $x[n] = u[n-2]$ into the system with $h[n] = \delta[n] - \delta[n-1]$, then the output is $y[n] = x[n] * h[n]$
 - The corresponding z-transform

$$X(z) = \frac{z^{-2}}{1 - z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$H(z) = 1 - z^{-1}, \quad \text{ROC: } z \neq 0$$

$$Y(z) = X(z)H(z) = z^{-2} \quad \text{ROC: } \mathbb{C} - \{0\}$$

- NOTE:** ROC after conv. is “bigger” than intersection of \mathcal{R}_x and \mathcal{R}_h .
- What is $y[n]$?** $y[n] = \delta[n-2]$.

Correlation

- Correlation:

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] \longleftrightarrow R_{xy}(z) = X(z)Y(z^{-1})$$

- Remind that

$$r_{xy}[l] = x[l] * y[-l]$$

- ROC:** contains **at least as much** of the z -plane as the intersection ROC of $X(z)$ with the ROC of $Y(z^{-1})$.

调制

$$g[n]h^*[n] \longleftrightarrow \frac{1}{2\pi j} \oint_C G(v)H^*(z^*/v^*)v^{-1}dv, \quad \text{ROC: } \mathcal{R}_g\mathcal{R}_h$$

Proof:

$$\begin{aligned} g[n]h^*[n] &\longleftrightarrow \sum_{n=-\infty}^{\infty} g[n]h^*[n]z^{-n} = \frac{1}{2\pi j} \sum_{k=-\infty}^{\infty} \left(\oint_C G(v)v^{n-1}dv \right) h^*[n]z^{-n} \\ &= \frac{1}{2\pi j} \oint_C G(v) \left(\sum_{k=-\infty}^{\infty} h^*[n]z^{-n}v^{n-1} \right) dv \\ &= \frac{1}{2\pi j} \oint_C G(v) \left(\sum_{k=-\infty}^{\infty} h^*[n](z/v)^{-n} \right) v^{-1}dv \\ &= \frac{1}{2\pi j} \oint_C G(v)H^*(z^*/v^*)v^{-1}dv \end{aligned}$$

Parseval's 定理

$$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v)H^*(1/v^*)v^{-1}dv, \quad \text{ROC: } \mathcal{R}_g\mathcal{R}_h$$

如果收敛域包括单位圆，那么令 $v = e^{j\omega}$

$$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega,$$

与 DTFT 的 Parseval 定理一致。

例子：指数序列的能量

例 (指数序列的能量)

计算序列 $x[n] = \alpha^n \mu[n]$, $0 < \alpha < 1$ 的能量。

根据 Parseval 定理,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi j} \oint_C X(z)X(z^{-1})z^{-1}dz = \sum_k \{Res(X(z)X(z^{-1})z^{-1}, z_k)\}$$

序列 $x[n]$ 收敛域 $|z| > \alpha$ 包含单位圆, 那么积分曲线 C 可以选单位圆。
指数序列的 Z 变换为

$X(z) = \frac{1}{1 - \alpha z^{-1}} \Rightarrow X(z)X(z^{-1})z^{-1} = \frac{1}{(z - \alpha)(1 - \alpha z)}$, 曲线 C 中的只有极点 $z = \alpha$, 那么留数计算结果为

$$\sum_k \{Res(X(z)X(z^{-1})z^{-1}, z_k)\} = \frac{1}{1 - \alpha z} \Big|_{z=\alpha} = \frac{1}{1 - \alpha^2}$$

利用 Z 变换计算线性卷积

长度为 $L+1$ 的序列 $\{x[n]\}, 0 \leq n \leq L$, 其 Z 变换为

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots + x[L]z^{-L}$$

长度为 $M+1$ 的序列 $\{h[n]\}, 0 \leq n \leq M$, 其 Z 变换为

$$H(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots + x[M]z^{-M}$$

Z 域乘积:

$$Y(z) = X(z)H(z) = y[0] + y[1]z^{-1} + y[2]z^{-2} + \dots + y[L+M]z^{-(L+M)}$$

Z 变换卷积定理:

$$y[n] = x[n] \otimes h[n] \longleftrightarrow Y(z) = X(z)H(z)$$

例子：多项式相乘计算线性卷积

例 (多项式相乘计算线性卷积)

计算序列 $x[n] = \{-2, 0, 1, -1, 3\}$ 和序列 $h[n] = \{1, 2, -1\}$ 的线性卷积 $y[n] = x[n] \otimes h[n]$ 。

$$X(z) = -2 + z^{-2} - z^{-3} + 3z^{-4} \quad H(z) = 1 + 2z^{-1} - z^{-2}$$

计算 $Y(z) = X(z)H(z)$:

$$Y(z) = -2 - 4z^{-1} + 3z^{-2} + z^{-3} + 7z^{-5} - 3z^{-6}$$

逆 Z 变换得到

$$y[n] = \{-2, -4, 3, 1, 0, 7, -3\}$$

利用 Z 变换计算圆周卷积

- 两个长度为 N 的序列 $x[n]$ 和 $h[n]$

- 圆周卷积:

$$y_C[n] = x[n] \otimes h[n] = \sum_{m=0}^{N-1} x[m] h[\langle n-m \rangle_N]$$

- 线性卷积:

$$y_L[n] = x[n] \otimes h[n] = \sum_{m=0}^{2N-1} x[m] h[n-m]$$

- 二者关系:

$$\begin{aligned} y_C[n] &= \sum_{m=0}^{N-1} x[m] h[\langle n-m \rangle_N] = \underbrace{\sum_{m=0}^n x[m] h[n-m]}_{n-m < 0, \text{ if } m > n} + \underbrace{\sum_{m=n+1}^{N-1} x[m] h[n+N-m]}_{n+N-m > N-1, \text{ if } m < n+1} \\ &= \sum_{m=0}^{2N-1} x[m] h[n-m] + \sum_{m=0}^{2N-1} x[m] h[n+N-m] \\ &= y_L[n] + y_L[n+N] \end{aligned}$$

利用 Z 变换计算圆周卷积

$$\begin{aligned} Y_C(z) &= \sum_{n=0}^{N-1} y_C[n] z^{-n} \\ &= \sum_{n=0}^{N-1} (y_L[n] + y_L[n+N]) z^{-n} \\ Y_L(z) &= \sum_{n=0}^{2N-2} y_L[n] z^{-n} \\ Y_C(z) &= \langle Y_L(z) \rangle_{z^{-N}-1} \end{aligned}$$

例子：利用 Z 变换计算圆周卷积

例 (利用 Z 变换计算圆周卷积)

长度为 4 的两个序列 $x[n] = \{1, 2, 3, 4\}$ 和 $h[n] = \{-1, 1, 0, 3\}$ 的圆周卷积 $y_C[n] = x[n] \circledast h[n]$ 。

① 线性卷积：

$$y_L[n] = x[n] \otimes h[n] = \{-1, -1, -1, 2, 10, 9, 12\}$$

② 圆周卷积：

$$y_C[n] = y_L[n] + y_L[n+N] = \{9, 8, 11, 2\}$$

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LTI: Constant-Coefficient Difference Equation

LTI system can be described via CCDE:

$$\sum_{l=0}^N d_l y[n-l] = \sum_{l=0}^M p_l x[n-l]$$

LTI: Constant-Coefficient Difference Equation

LTI system can be described via CCDE:

$$\sum_{l=0}^N d_l y[n-l] = \sum_{l=0}^M p_l x[n-l]$$

applying z-transform both sides

$$Y(z) \sum_{l=0}^N d_l z^{-l} = X(z) \sum_{l=0}^M p_l z^{-l}$$

LTI: Constant-Coefficient Difference Equation

LTI system can be described via CCDE:

$$\sum_{l=0}^N d_l y[n-l] = \sum_{l=0}^M p_l x[n-l]$$

applying z-transform both sides

$$Y(z) \sum_{l=0}^N d_l z^{-l} = X(z) \sum_{l=0}^M p_l z^{-l}$$

transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^M p_l z^{-l}}{\sum_{l=0}^N d_l z^{-l}}$$

LTI: Constant-Coefficient Difference Equation

LTI system can be described via CCDE:

$$\sum_{l=0}^N d_l y[n-l] = \sum_{l=0}^M p_l x[n-l]$$

applying z-transform both sides

$$Y(z) \sum_{l=0}^N d_l z^{-l} = X(z) \sum_{l=0}^M p_l z^{-l}$$

transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^M p_l z^{-l}}{\sum_{l=0}^N d_l z^{-l}}$$

normally, $H(z)$ are rational functions of z^{-1}

Rational z-transforms

$$H(z) = \frac{\sum_{l=0}^M p_l z^{-l}}{\sum_{l=0}^N d_l z^{-l}}$$

alternative representation as rational functions of z

$$H(z) = z^{(N-M)} \frac{\sum_{l=0}^M p_l z^{M-l}}{\sum_{l=0}^N d_l z^{N-l}}$$

or factorized version

$$H(z) = \frac{p_0 \prod_{l=1}^M (1 - \varepsilon_l z^{-1})}{d_0 \prod_{l=1}^N (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{l=1}^M (z - \varepsilon_l)}{d_0 \prod_{l=1}^N (z - \lambda_l)}$$

Zeros and Poles

$$H(z) = \frac{p_0 \prod_{l=1}^M (1 - \varepsilon_l z^{-1})}{d_0 \prod_{l=1}^N (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{l=1}^M (z - \varepsilon_l)}{d_0 \prod_{l=1}^N (z - \lambda_l)}$$

Zeros: roots $z = \varepsilon_l$ of the numerator polynomial, when $H(z) = 0$.

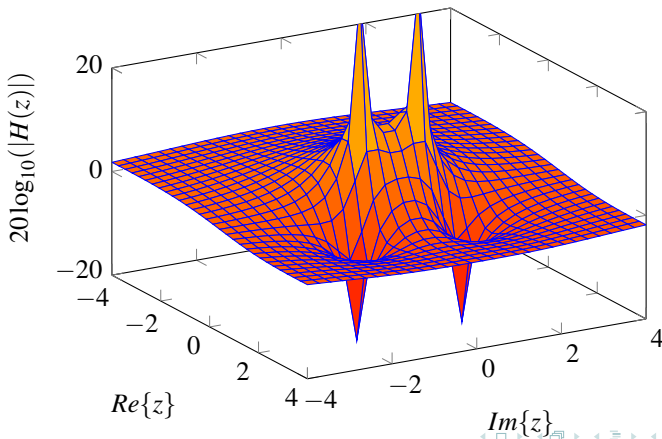
Poles: roots $z = \lambda_l$ of the denominator polynomial, when $H(z) \rightarrow \infty$.

- if $N > M$, there are additional $N - M$ zeros at $z = 0$
- if $N < M$, there are additional $M - N$ poles at $z = 0$

Example: $M=N$

$$H(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

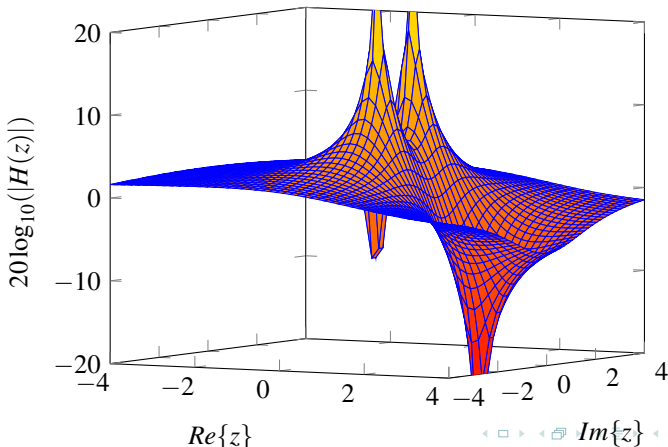
Zeros: $z = 1.2 \pm j1.2$, **Poles:** $z = 0.4 \pm j0.6928$



Example: $M > N$

$$H(z) = \frac{1 - 2.4z^{-1}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

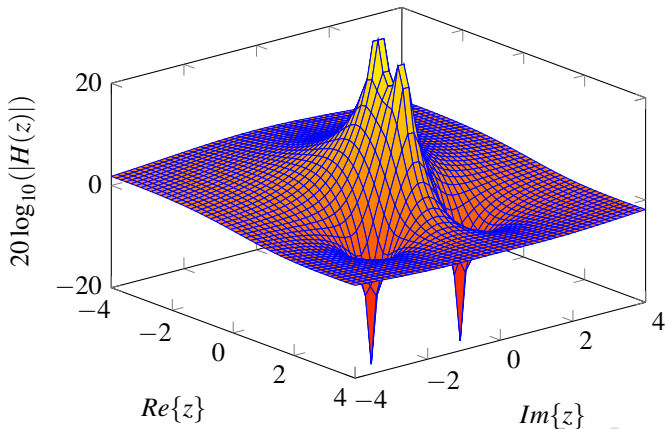
Zeros: $z \in \{2.4, 0\}$, **Poles:** $z = 0.4 \pm j0.6928$



Example: $M < N$

$$H(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1}}$$

Zeros: $z = 1.2 \pm j1.2$, **Poles:** $z \in \{0.8, 0\}$

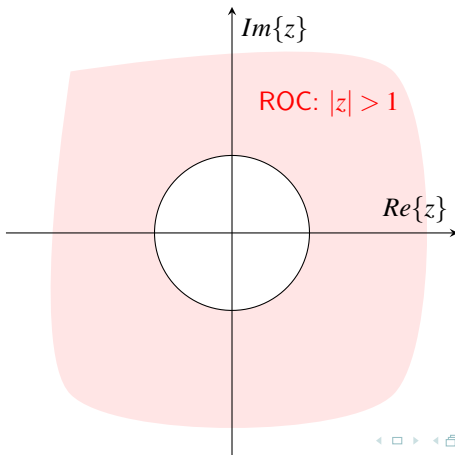


z-domain

The z-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

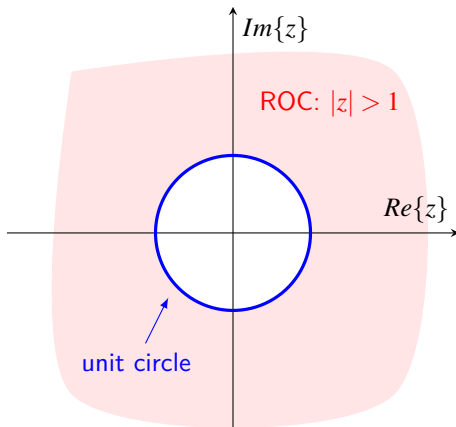


z-domain

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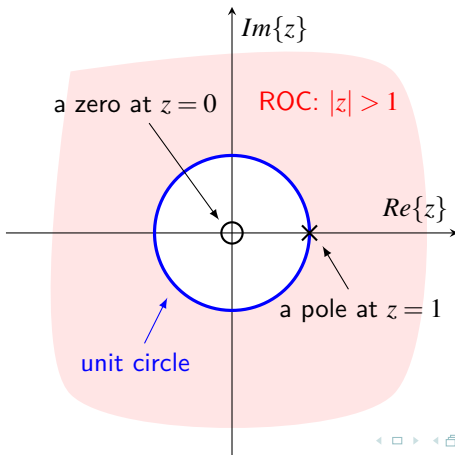


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$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

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ROC, Poles and Causality

Z-transform of **causal sequences**

$$\alpha^n \mu[n] \longleftrightarrow \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |z| > |\alpha|$$

with a pole at $z = \alpha$.

ROC, Poles and Causality

Z-transform of **causal sequences**

$$\alpha^n \mu[n] \longleftrightarrow \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |z| > |\alpha|$$

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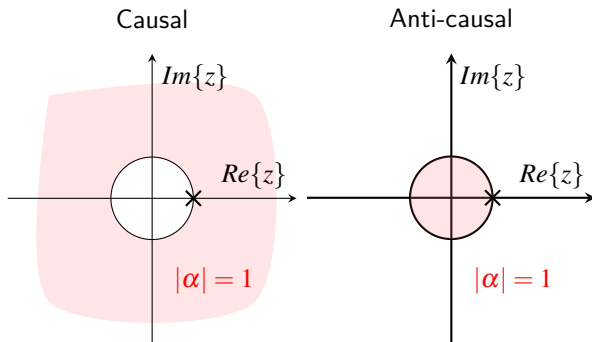
Z-transform of **anti-causal sequences**

$$-\alpha^n \mu[-n-1] \longleftrightarrow \frac{1}{1 - \alpha z^{-1}} \quad \text{ROC: } |z| < |\alpha|$$

with a pole at $z = \alpha$.

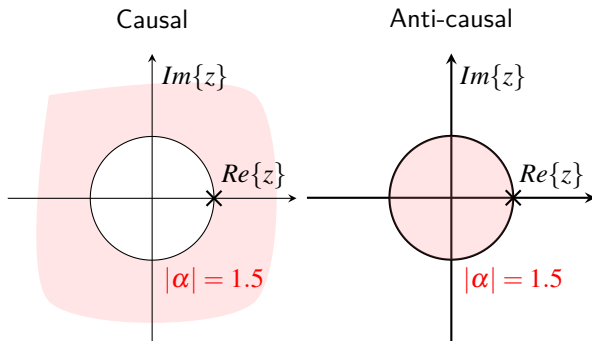
ROC, Poles and Causality

- poles determines the boundary of ROC
- causality determines whether ROC is interior or exterior of the boundary circle



ROC, Poles and Causality

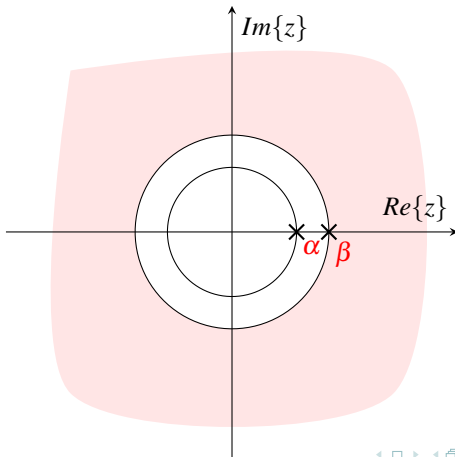
- poles determines the boundary of ROC
- causality determines whether ROC is interior or exterior of the boundary circle



Multiple poles: causal sequence

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[n - N_0]$$

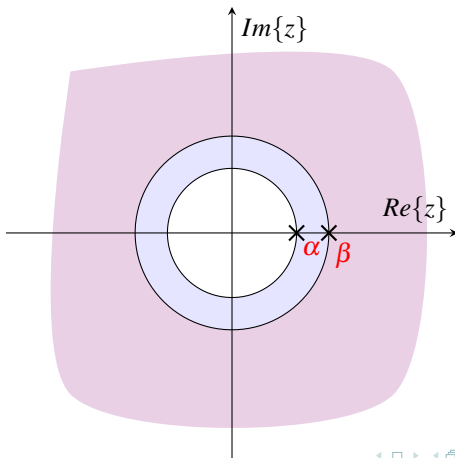
two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



Multiple poles: causal sequence

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[n - N_0]$$

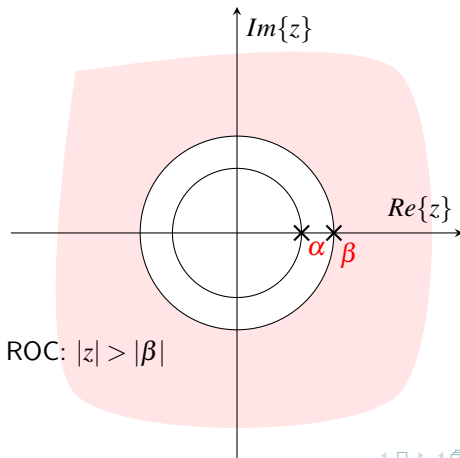
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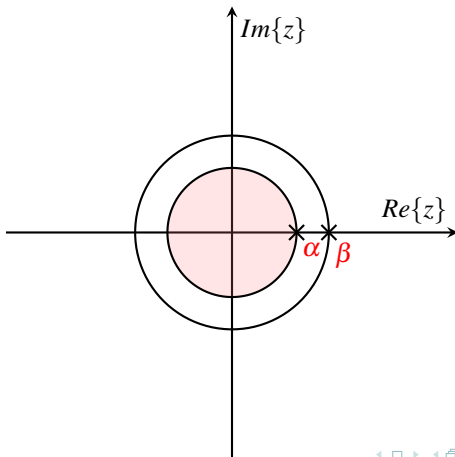
two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



Multiple poles: anti-causal sequence

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[-n - N_0]$$

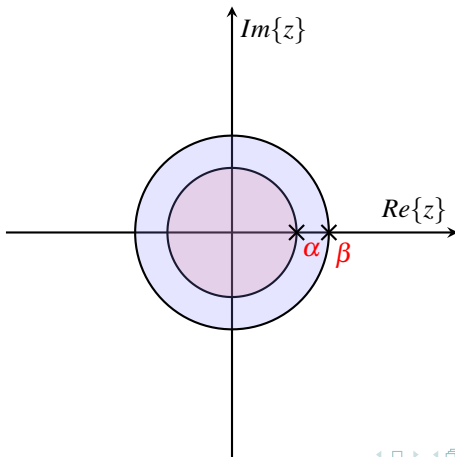
two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



Multiple poles: anti-causal sequence

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[-n - N_0]$$

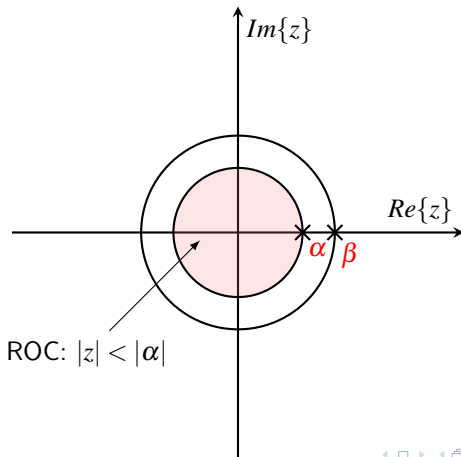
two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



Multiple poles: anti-causal sequence

$$x[n] = (r_1 \alpha^n + r_2 \beta^n) \mu[-n - N_0]$$

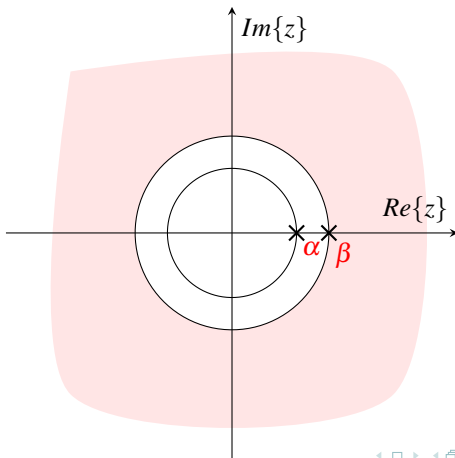
two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



Multiple poles: two-sided sequence

$$x[n] = r_1 \alpha^n \mu[-n - N_0] + r_2 \beta^n \mu[n - N_0]$$

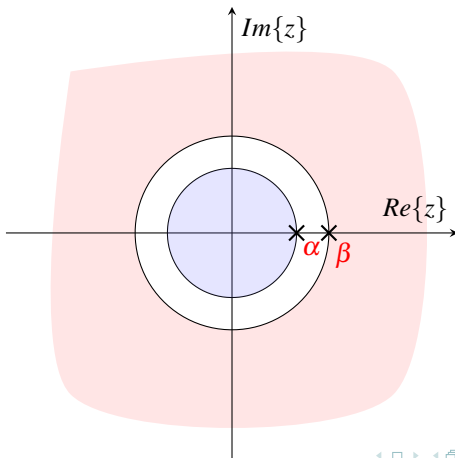
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Multiple poles: two-sided sequence

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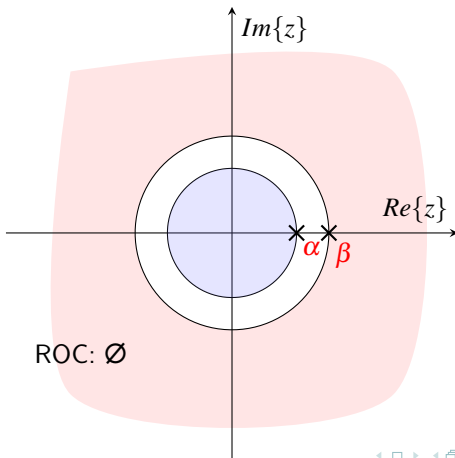
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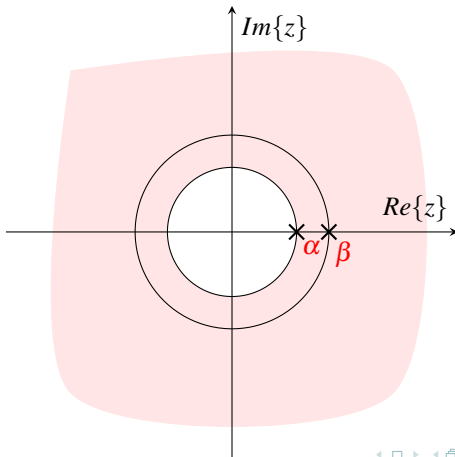
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Multiple poles: two-sided sequence

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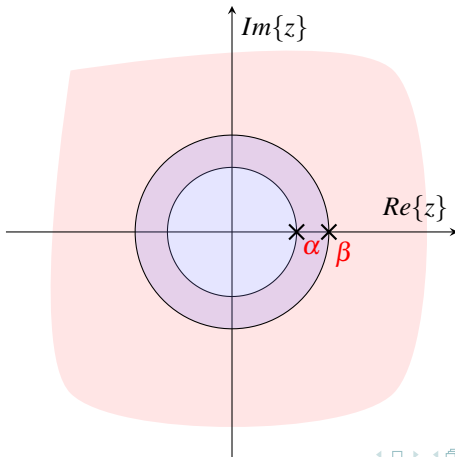
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Multiple poles: two-sided sequence

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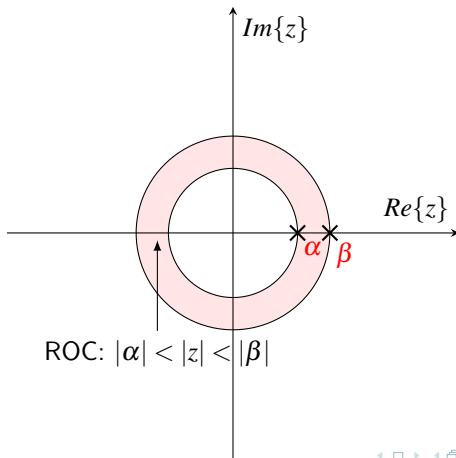
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Multiple poles: two-sided sequence

$$x[n] = r_1 \alpha^n \mu[n - N_0] + r_2 \beta^n \mu[-n - N_0]$$

two poles $z = \alpha$ and $z = \beta$, $|\alpha| < |\beta|$



z-transform with Matlab

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

- find the zeros and poles
- determine the ROC

Zeros and Poles with Matlab

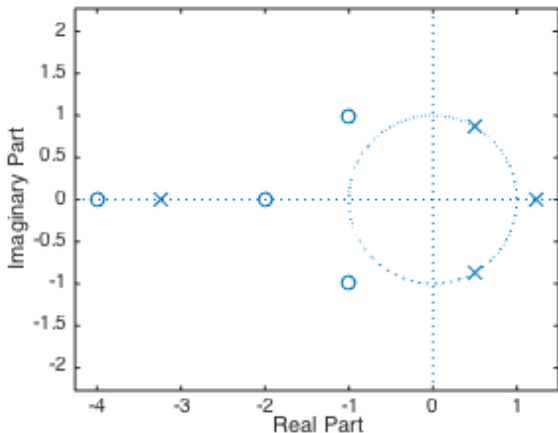
tf2zp(num,den)

Transfer function to zero-pole conversion.

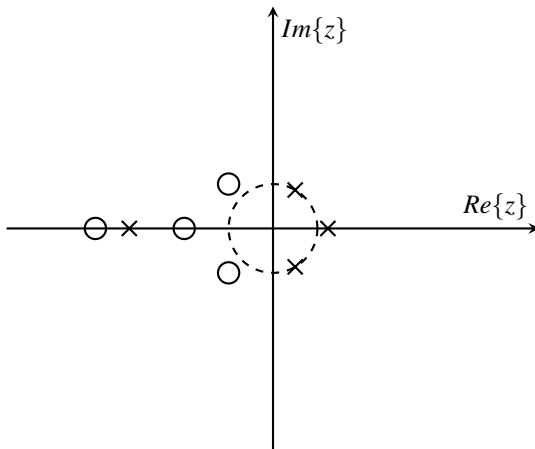
- *num*: numerator of transfer function
- *den*: denominator of transfer function

Zeros and Poles with Matlab

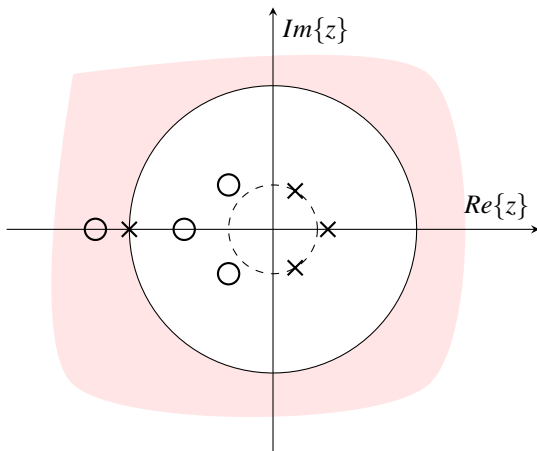
`zplane(num,den)`



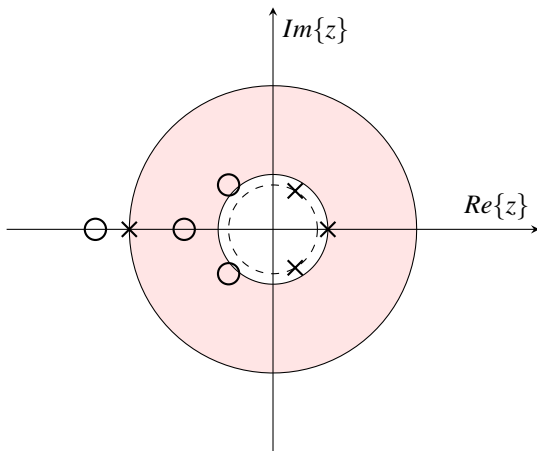
ROC



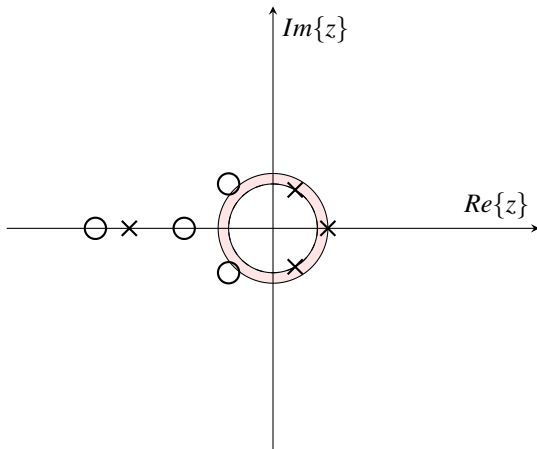
ROC



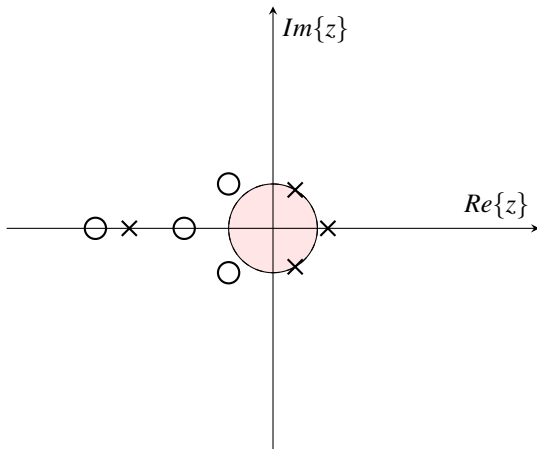
ROC



ROC



ROC



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逆 Z 变换

Z 变换:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

逆 Z 变换:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz, \quad C \text{ 为收敛域中的任意闭合曲线}$$

逆 Z 变换

序列 $x[n]$ 的 Z 变换 $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, 两边乘上 z^{n-1} ,

$$z^{n-1}X(z) = z^{n-1} \sum_{l=-\infty}^{\infty} x[l]z^{-l}$$

两边同时沿闭合曲线 C 逆时针求积分

$$\oint_C X(z)z^{n-1}dz = \oint_C z^{n-1} \sum_{l=-\infty}^{\infty} x[l]z^{-l}dz = \sum_{l=-\infty}^{\infty} x[l] \underbrace{\oint_C z^{n-1-l}dz}_{=2\pi j\delta[n-l]} = 2\pi jx[n]$$

求解逆 Z 变换的方法

- ① 留数法
- ② 观察法：通过查表得到逆 Z 变换
- ③ 部分分式法
- ④ 幂级数展开法

留数法

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

柯西留数定理：

$$x[n] = \sum_{k=0}^N \text{Res}(X(z) z^{n-1}, \lambda_k)$$

λ_k 为 $X(z)z^{n-1}$ 的 L 重极点， Res 为留数：

$$\text{Res}(f(z), \lambda_k) = \frac{1}{(L-1)!} \left[\frac{d^{L-1}}{dz^{L-1}} f(z) (z - \lambda_k)^L \right]_{z=\lambda_k}$$

例子：留数法

计算 Z 变换为 $X(z)$ 的序列：

$$X(z) = \frac{z}{(z-1)^2}, \quad |z| > 1$$

$$f(z) = X(z)z^{n-1} = \frac{z^n}{(z-1)^2}$$

该序列具有：

- 一个 2 重极点 $z=1$, $\forall n$;
- 一个 $-n$ 重极点 $z=0$, $\forall n < 0$

例子：留数法

极点 $z = 1$ 处的留数为 ($L = 2$):

$$\text{Res}(f(z), 1) = \left[\frac{d}{dz} f(z) (z-1)^2 \right]_{z=\lambda_k} = \frac{dz^n}{dz} \Big|_{z=\lambda_k} = n z^{n-1} \Big|_{z=1} = n$$

极点 $z = 0$ 处的留数 $L = -n, n < 0$:

$$\begin{aligned} \text{Res}(f(z), 0) &= \frac{1}{(-n-1)!} \frac{d^{-n-1}}{dz^{-n-1}} f(z) z^{-n} \Big|_{z=0} \\ &= \frac{1}{(-n-1)!} \frac{d^{-n-1}}{dz^{-n-1}} \left(\frac{1}{(z-1)^2} \right) \Big|_{z=0} \\ &= \frac{(-n)!}{(-n-1)!} = -n \end{aligned}$$

z -逆变换为:

$$x[n] = \text{Res}(f(z), 1) + \text{Res}(f(z), 0) = n\mu[n]$$

部分分式法

- λ_k 为 $X(z)$ 的单极点,

$$X(z) = \sum_{k=1}^N \frac{\rho_k}{1 - \lambda_k z^{-1}}$$

- 当收敛域 ROC 为 $|z| > \max\{\lambda_k\}$:

$$x[n] = \sum_{k=1}^N \rho_k \lambda_k^n \mu[n]$$

- 当收敛域 ROC 为 $|z| < \min\{\lambda_k\}$:

$$x[n] = - \sum_{k=1}^N \rho_k \lambda_k^n \mu[-n-1]$$

- v 为 $X(z)$ 的 L 重极点 (ROC: $|z| > |v|$),

$$X(z) = \sum_{k=1}^L \frac{\gamma_k}{(1 - v z^{-1})^k}$$

$$x[n] = \sum_{k=1}^L (n+k-1)! \gamma_k v^n \mu[n]$$

部分分式法

任意的有理 Z 变换可以写成：

$$X(z) = \frac{P(z)}{D(z)} = \frac{\sum_{l=0}^M p_l z^{-l}}{\sum_{l=0}^N d_l z^{-l}}$$

当 $M \geq N$ ，我们可以将其写成如下形式

$$X(z) = \underbrace{\sum_{l=0}^{M-N} \eta_l z^{-l}}_{\text{有限长序列的 } Z \text{ 变换}} + \underbrace{\frac{P_1(z)}{D(z)}}_{\text{真分式}}$$

η_l 可以通过长除法得到。

真分式的逆 Z 变换

假设 $X(z)$ 为真分式，那么可以通过因式分解得到如下形式，

$$X(z) = \underbrace{\sum_{k=1}^{N-L} \frac{\rho_k}{1 - \lambda_k z^{-1}}}_{\text{单极点分式}} + \underbrace{\sum_{k=1}^L \frac{\gamma_k}{(1 - \nu z^{-1})^k}}_{\text{多极点分式}}$$

两种分式的分子分别为

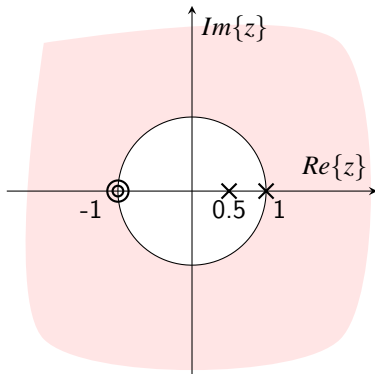
$$\rho_k = \text{Res}(X(z), \lambda_k) = (1 - \lambda_k z^{-1})X(z)|_{z=\lambda_k},$$

$$\gamma_k = \frac{1}{(L-k)!(-\nu)^{L-k}} \frac{d^{L-k}}{d(z^{-1})^{L-k}} [(1 - \nu z^{-1})^L X(z)]|_{z=\nu}$$

例子：真分式的逆 Z 变换

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - 0.5z^{-1})(1 - z^{-1})}, \quad |z| > 1$$

其零极点图为



例子：真分式的逆 Z 变换

$$X(z) = \frac{(1+z^{-1})^2}{(1-0.5z^{-1})(1-z^{-1})} = \underbrace{B_0}_{B_0\delta[n]} + \underbrace{\frac{A_1}{1-0.5z^{-1}}}_{A_1(0.5)^n\mu[n]} + \underbrace{\frac{A_2}{1-z^{-1}}}_{A_2\mu[n]}$$

长除法：

$$B_0 = 2$$

单极点留数：

$$A_1 = [X(z)(1-0.5z^{-1})]_{z=0.5} = -9$$

$$A_2 = [X(z)(1-z^{-1})]_{z=1} = 8$$

逆 Z 变换后的序列为：

$$x[n] = 2\delta[n] - \frac{9}{2^n}\mu[n] + 8\mu[n]$$

利用 Matlab 进行部分分式展开

$$X(z) = \frac{18z^3}{18z^3 + 3z^2 - 4z - 1}$$

$$[r,p,k]=\text{residuez}(\text{num},\text{den})$$

- r 为留数 [0.3600, 0.2400, 0.4000]
- p 为极点 [0.5000, -0.3333, -0.3333]
- k 为常数 []

那么这个 Z 变换表达式可以写成

$$X(z) = \frac{0.36}{1 - 0.5z^{-1}} + \frac{0.24}{1 + 1/3z^{-1}} + \frac{0.4}{(1 + 1/3z^{-1})^2}$$

幂级数展开法

Taylor 级数展开: $X(z) = \log(1 + az^{-1}), \quad |z| > |a|$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n} \longleftrightarrow x[n] = \begin{cases} (-1)^{n+1} \frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

长除法: $X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$

$$\begin{array}{r} 1 - az^{-1} \overline{) 1 + az^{-1} + a^2 z^{-2} + \dots} \\ \underline{1} \phantom{+ az^{-1} + a^2 z^{-2} + \dots} \\ az^{-1} \phantom{+ a^2 z^{-2} + \dots} \\ \underline{az^{-1} - a^2 z^{-2}} \\ a^2 z^{-2} \dots \end{array}$$

$$X(z) = 1 + az^{-1} + a^2 z^{-2} + \dots \longleftrightarrow x[n] = a^n \mu[n]$$

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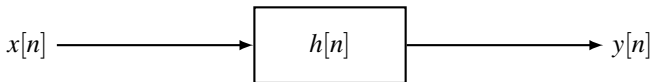
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LTI 离散时间系统的传输函数



系统输出:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

卷积定理: 令 $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, 那么

$$Y(z) = X(z)H(z)$$

传输函数:

- ① 传输函数 $H(z)$ 为冲激响应的 Z 变换;
- ② 传输函数 $H(z)$ 为输入序列 Z 变换与输出序列 Z 变换的比值。

Z 变换与 LTI 离散时间系统

- **LTI 系统：** 可以用线性常微分方程来描述

$$\sum_{l=0}^N d_l y[n-l] = \sum_{l=0}^M p_l x[n-l]$$

- **Z 变换：** 两边同时求 Z-变换得到

$$Y(z) \sum_{l=0}^N d_l z^{-l} = X(z) \sum_{l=0}^M p_l z^{-l}$$

- **传输函数：**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^M p_l z^{-l}}{\sum_{l=0}^N d_l z^{-l}} = \frac{p_0 \prod_{l=1}^M (1 - \varepsilon_l z^{-1})}{d_0 \prod_{l=1}^N (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{l=1}^M (z - \varepsilon_l)}{d_0 \prod_{l=1}^N (z - \lambda_l)}$$

零点与极点

$$H(z) = \frac{p_0 \prod_{l=1}^M (1 - \varepsilon_l z^{-1})}{d_0 \prod_{l=1}^N (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0 \prod_{l=1}^M (z - \varepsilon_l)}{d_0 \prod_{l=1}^N (z - \lambda_l)}$$

零点： 当 $H(z) = 0$ 时，分子多项式的根 $z = \varepsilon_l$ ；

极点： 当 $H(z) \rightarrow \infty$ 时，分母多项式的根 $z = \lambda_l$

NOTES

- $N > M$, 需要增加额外 $N - M$ 个在 $z = 0$ 处的零点；
- $N < M$, 需要增加额外 $M - N$ 个在 $z = 0$ 处的极点。

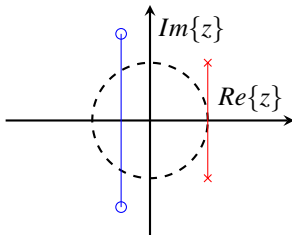
实系统的零点和极点

实际系统的冲激响应序列都是实数，即 $h[n] \in \mathbb{R}, \forall n$ 。

实系统的零点和极点

实系统的零点和极点都是关于实轴对称的。

$$H(z) = \frac{1 + z^{-1} + 2.5z^{-2}}{1 - 2z^{-1} + 2z^{-2}} = \frac{[1 + (1 - 3i)/2z^{-1}][1 - (1 + 3i)/2z^{-1}]}{[1 - (1 + i)z^{-1}][1 - (1 - i)z^{-1}]}$$



例子：FIR 系统

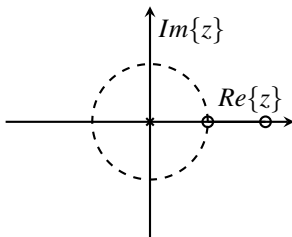
FIR 系统： 定义 FIR 系统冲激响应的 Z 变换为

$$H(z) = (1 - z^{-1})(1 - 2z^{-1})$$

零点与极点： 冲激响应的 Z 变换可以写成

$$H(z) = \frac{(z-1)(z-2)}{z^2}$$

系统有两个零点 $z_1 = 1, z_2 = 2$ ，原点 $z = 0$ 处存在多重极点



例子：IIR 系统

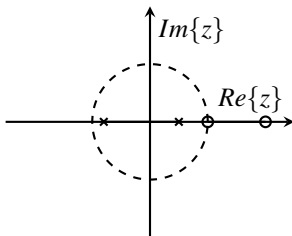
IIR 系统： 定义 IIR 系统冲激响应的 Z 变换为

$$H(z) = \frac{(1 - z^{-1})(1 - 2z^{-1})}{(1 - 0.5z^{-1})(1 + 0.8z^{-1})} \quad |z| > 0.8$$

零点与极点： 冲激响应的 Z 变换可以写成

$$H(z) = \frac{(z - 1)(z - 2)}{(z - 0.5)(z - 0.8)}$$

系统有两个零点 $z_1 = 1, z_2 = 2$ ，两个极点 $z = 0.5, z = 0.8$



Outline

- 1 z-Transform
 - Definition
 - Property of z-Transform
 - Rational z-Transform
 - Inverse z-Transform

- 2 LTI System in Transfer Domain
 - Transfer Function
 - Analysis of LTI

回顾：系统的稳定性和因果性

稳定性： 冲激响应 $h[n]$ 绝对可和

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

因果性： 冲激响应 $h[n]$ 满足

$$h[n] = 0, \forall n < 0$$

系统的因果性判定：Z 域分析

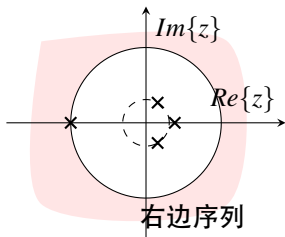
- 因果 LTI 系统的冲激响应 $h[n]$ 满足

$$h[n] = 0, \forall n < 0$$

- 系统的冲激响应是右边序列
- 右边序列的收敛域为过最大幅度的极点的圆外

系统因果性判定：Z 域分析

冲激响应序列的 Z 变换的收敛域为过最大幅度极点的圆外时，判定该系统为因果 LTI 系统。



系统的稳定性判定：Z 域分析

- 稳定 LTI 系统的冲激响应 $h[n]$ 满足绝对可和条件（充分必要条件）

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- 冲激响应序列 $h[n]$ 的 **DTFT** 变换存在
- 冲激响应序列 $h[n]$ 的 Z 变换收敛域包括单位圆
- 如果是因果系统，冲激响应序列 $h[n]$ 的 Z 变换的所有极点都在单位圆内；如果是非因果系统，冲激响应序列 $h[n]$ 的 Z 变换的所有极点都在单位圆外；

系统稳定性判定：Z 域分析

因果系统的冲激响应序列的 Z 变换的所有极点都在单位圆内，判定该系统为稳定 LTI 系统。

非因果系统的冲激响应序列的 Z 变换的所有极点都在单位圆外，判定该系统为稳定 LTI 系统。

例子：Z 域分析系统稳定性

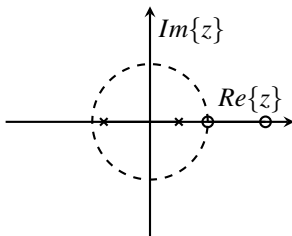
IIR 系统： 定义 IIR 系统冲激响应的 Z 变换为

$$H(z) = \frac{(1 - z^{-1})(1 - 2z^{-1})}{(1 - 0.5z^{-1})(1 + 0.8z^{-1})}$$

零点与极点： 冲激响应的 Z 变换可以写成

$$H(z) = \frac{(z - 1)(z - 2)}{(z - 0.5)(z - 0.8)}$$

系统有两个零点 $z_1 = 1, z_2 = 2$ ，两个极点 $z = 0.5, z = 0.8$



例子：Z 域分析系统稳定性

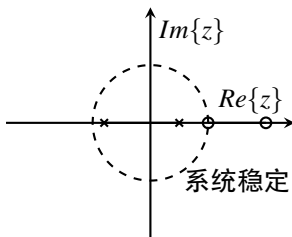
IIR 系统： 定义 IIR 系统冲激响应的 Z 变换为

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零点与极点： 冲激响应的 Z 变换可以写成

$$H(z) = \frac{(z - 1)(z - 2)}{(z - 0.5)(z - 0.8)}$$

系统有两个零点 $z_1 = 1, z_2 = 2$ ，两个极点 $z = 0.5, z = 0.8$



系统的频率响应

DTFT 变换与 Z 变换之间的关系:

$$\begin{aligned} H(e^{j\omega}) &= H(z)|_{z=e^{j\omega}} \\ &= e^{j\omega(N-M)} \frac{p_0 \prod_{l=1}^M (e^{j\omega} - \varepsilon_l)}{d_0 \prod_{l=1}^N (e^{j\omega} - \lambda_l)} \end{aligned}$$

- 幅度响应:

$$|H(e^{j\omega})| = \left| \frac{p_0}{d_0} \right| \frac{\prod_{l=1}^M |e^{j\omega} - \varepsilon_l|}{\prod_{l=1}^N |e^{j\omega} - \lambda_l|}$$

- 相位响应:

$$\begin{aligned} \theta(\omega) &= \arg H(e^{j\omega}) \\ &= (N-M)\omega + \arg(p_0/d_0) + \sum_{l=1}^M \arg(e^{j\omega} - \varepsilon_l) - \sum_{l=1}^N \arg(e^{j\omega} - \lambda_l) \end{aligned}$$

实系统的幅度响应

根据实序列 DTFT 变换的对称性质

$$H^*(e^{j\omega}) = H(e^{-j\omega})$$

Z 域中的关系为

$$H^*(z) = H(z^{-1})|_{z=e^{j\omega}}$$

实系统的幅度响应

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) \\ &= |p_0/d_0|^2 \frac{\prod_{l=1}^M (e^{j\omega} - \varepsilon_l)(e^{-j\omega} - \varepsilon_l^*)}{\prod_{l=1}^N (e^{j\omega} - \lambda_l)(e^{-j\omega} - \lambda_l^*)} \end{aligned}$$

实系统的相位响应

根据 Z 变换和 DFTT 变换之间的关系

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = |H(e^{j\omega})|e^{j\theta(\omega)}$$

根据实序列 DFTFT 变换的对称性质

$$\underbrace{H(e^{-j\omega})}_{H(z^{-1})|_{z=e^{j\omega}}} = H^*(e^{j\omega}) = |H(e^{j\omega})|e^{-j\theta(\omega)}$$

那么系统的相位响应

$$\theta(\omega) = \arctan \left[\frac{H_{im}(z)}{H_{re}(z)} \right]_{z=e^{j\omega}} = \frac{1}{2j} \ln \left[\frac{H(z)}{H(z^{-1})} \right]_{z=e^{j\omega}}$$

实系统的群延迟

- 群延迟：

$$\tau_g(\omega) = \frac{d\theta(\omega)}{d\omega}$$

- 群延迟的 Z 域分析：

$$\tau_g(\omega) = \operatorname{Re} \left[z \frac{\ln H(z)}{dz} \right]_{z=e^{j\omega}}$$

- Proof:

$$\begin{aligned} -j \frac{d[\ln H(e^{j\omega})]}{d\omega} &= -j \frac{d[\ln H(z)]}{dz} \frac{dz}{d\omega} \Big|_{z=e^{j\omega}} = \frac{d[\ln H(z)]}{dz} z \Big|_{z=e^{j\omega}} \\ \ln H(e^{j\omega}) &= \ln |H(e^{j\omega})| + j\theta(\omega) \\ &\Rightarrow \\ -j \frac{d[\ln H(e^{j\omega})]}{d\omega} &= -j \frac{d[\ln |H(e^{j\omega})|]}{d\omega} + \frac{d\theta(\omega)}{d\omega} \end{aligned}$$

频率响应的几何描述

考虑一个基本单元：

$$v(\omega) = (e^{j\omega} - \rho e^{j\phi})$$

其中，如果这个单元在分子上，那么 $\rho e^{j\phi}$ 为零点；如果这个单元在分母上，那么 $\rho e^{j\phi}$ 为零点。