Digital Signal Processing

Module 3: from Euclid to Hilbert

Module Overview:

- Module 3.1: Signal processing as geometry or from Euclid to Hilbert spaces
- Module 3.2: Vectors, vector spaces, inner products, and Hilbert spaces
- Module 3.3: Bases for Hilbert spaces

Digital Signal Processing

Module 3.1: a tale of two (and more) vectors

Let's look at $\mathbb{R}^2$

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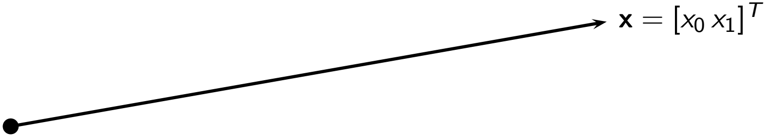

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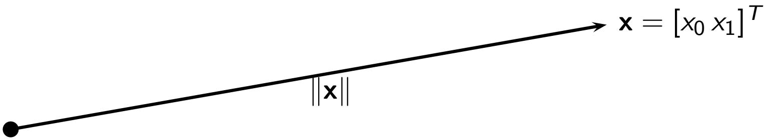

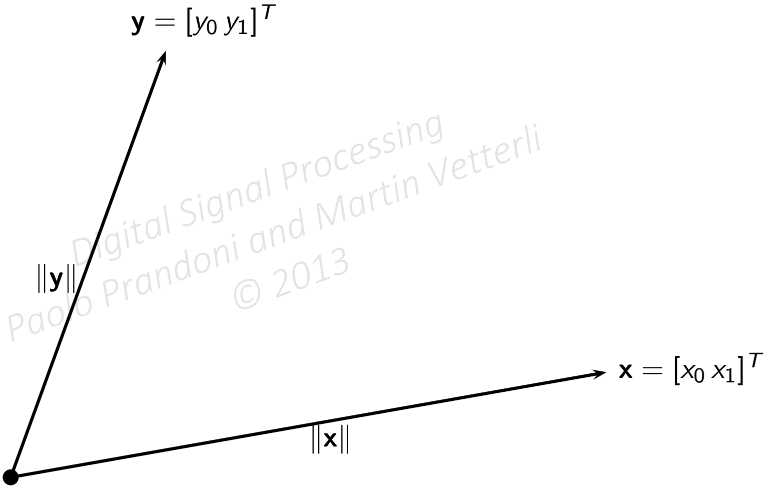

$\mathbf{y} = [y_0\ y_1]^T$
$\|\mathbf{y}\|$
$\mathbf{x} = [x_0\ x_1]^T$
$\|\mathbf{x}\|$

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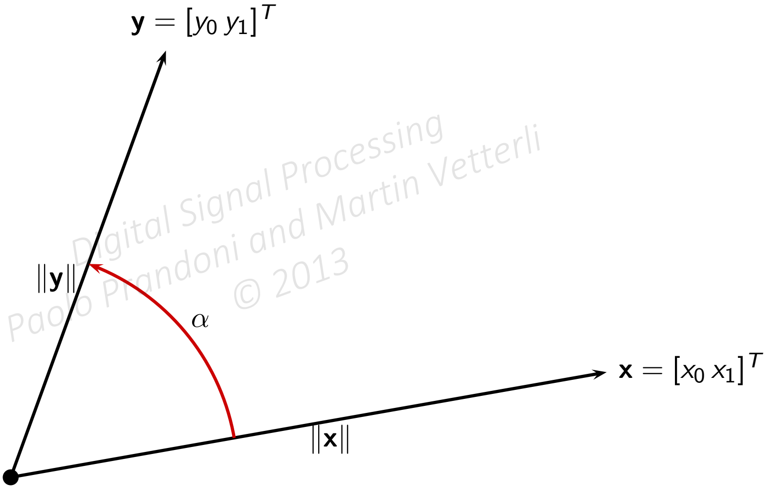

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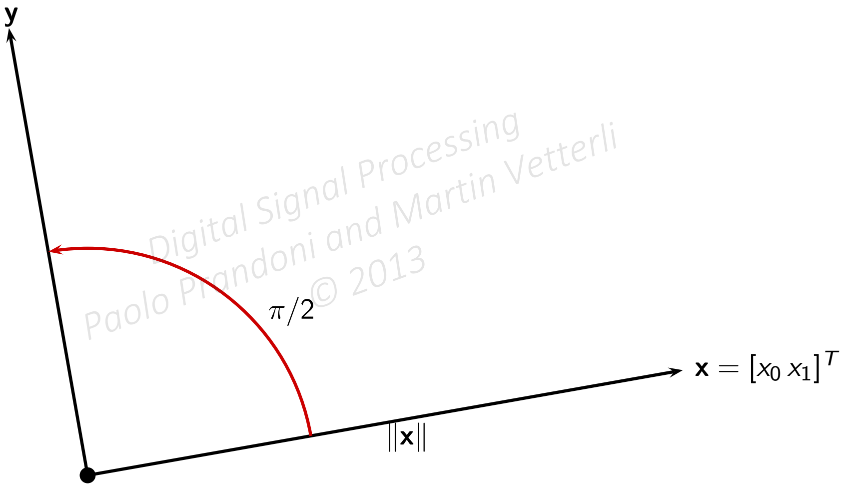
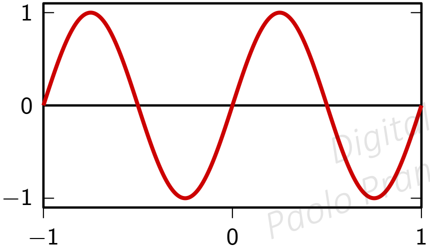
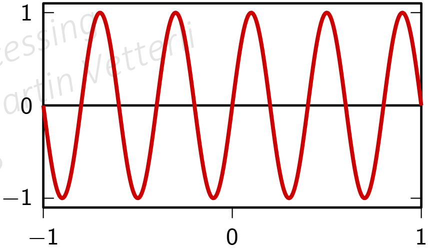


Vectors can be very general objects!

Example: space of square-integrable functions over $[-1,1]$: $L_2([-1,1])$



$\mathbf{x}^{(1)} = \sin(f_1\, t), \quad f_1 = 2\pi$

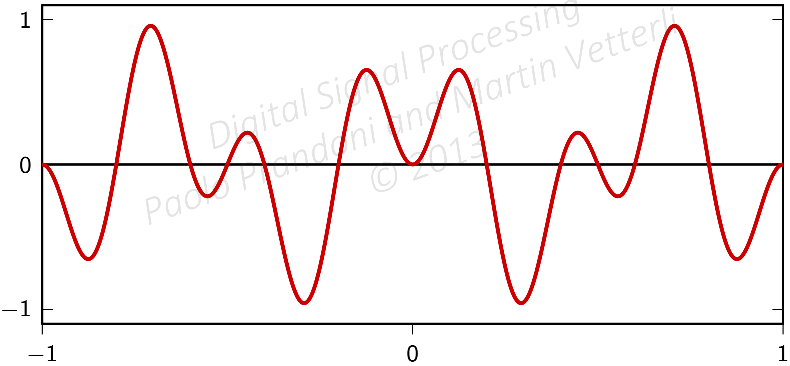


$\mathbf{x}^{(2)} = \sin(f_2\, t), \quad f_2 = 5\pi$

$$\langle \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \rangle = \int_{-1}^{1} \sin(f_1\, t) \sin(f_2\, t) dt$$

Orthogonality in a functional vector space.

$\mathbf{x}^{(1)} \perp \mathbf{x}^{(2)}$ if $f_1 \neq f_2$ and f_1, f_2 integer multiples of a fundamental (harmonically related)



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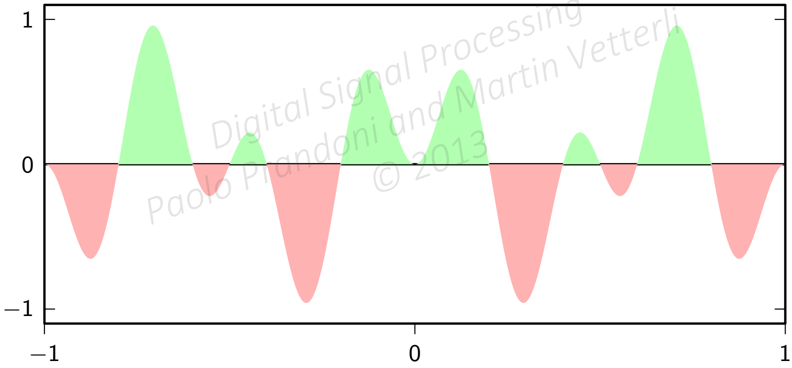

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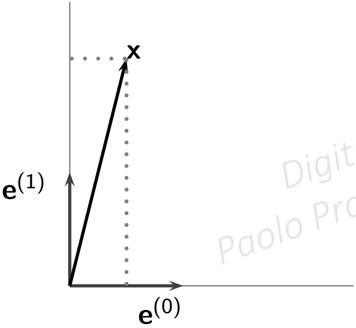
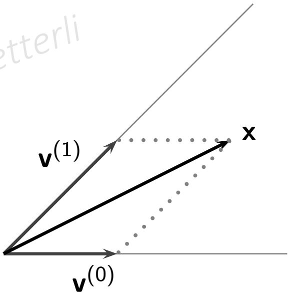
Vectors spanning a space

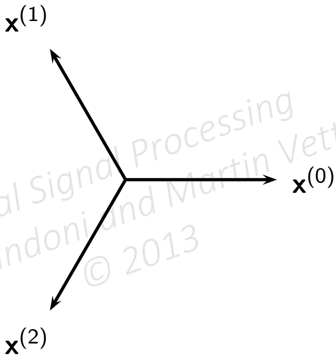


orthogonal basis



biorthogonal basis

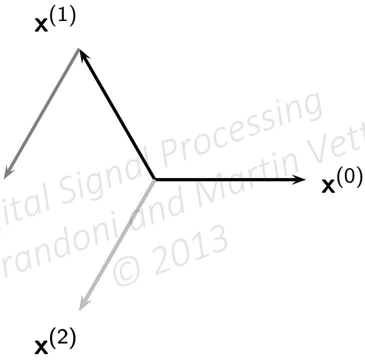
Too many vectors for the space



Linear dependence:

$$\exists\{a_0, a_1, a_2\} \text{ s.t. } a_0\mathbf{x}^{(0)} + a_1\mathbf{x}^{(1)} + a_2\mathbf{x}^{(2)} = 0$$

Too many vectors for the space



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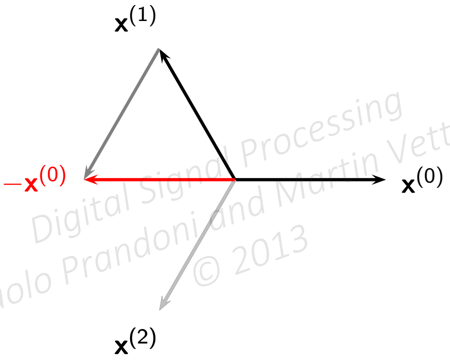

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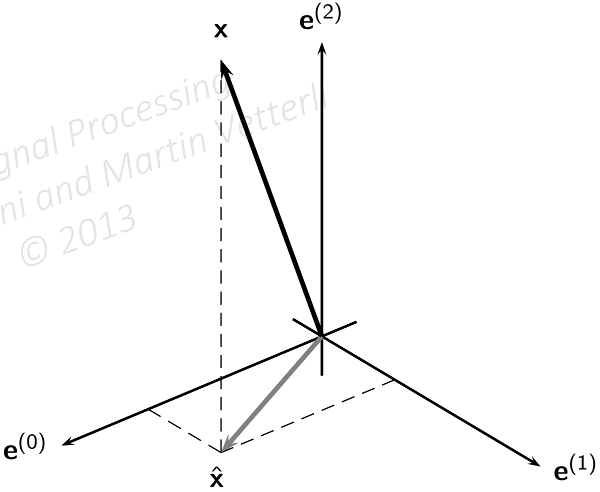
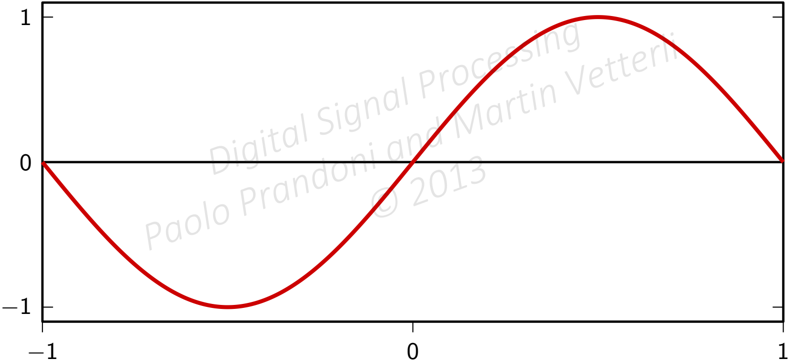
Not enough vectors for the space

subspace projection: $\hat{\mathbf{x}}$ is the closest approximation to $\mathbf{x}$ in the space spanned by $\{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$



Interesting questions when the space has ∞ dimensions

$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$



$N = 0$

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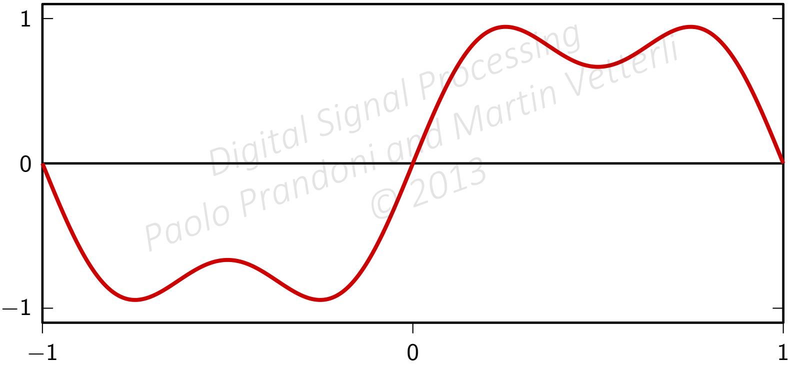
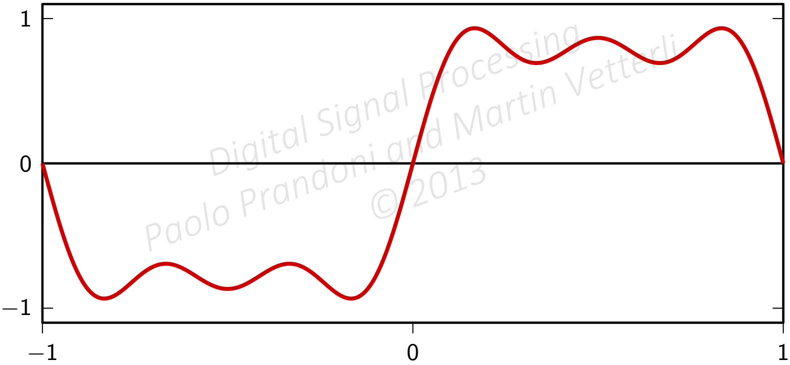


$N = 1$

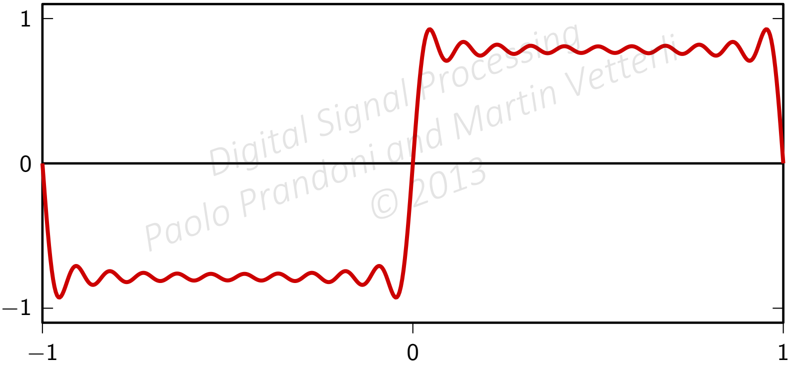
$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$



$N = 2$

Interesting questions when the space has ∞ dimensions

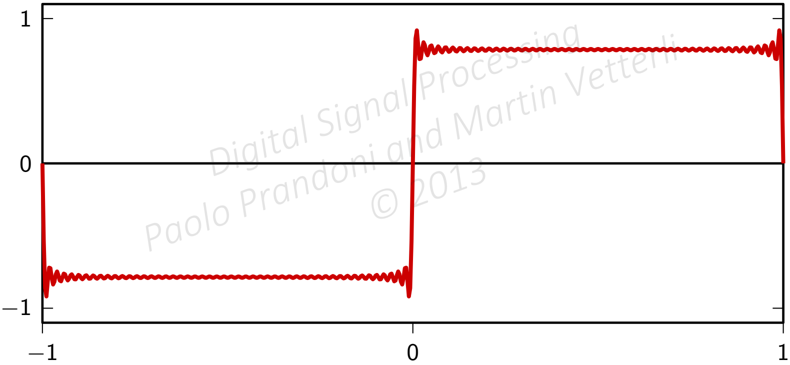
$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$



$N = 10$

Interesting questions when the space has ∞ dimensions

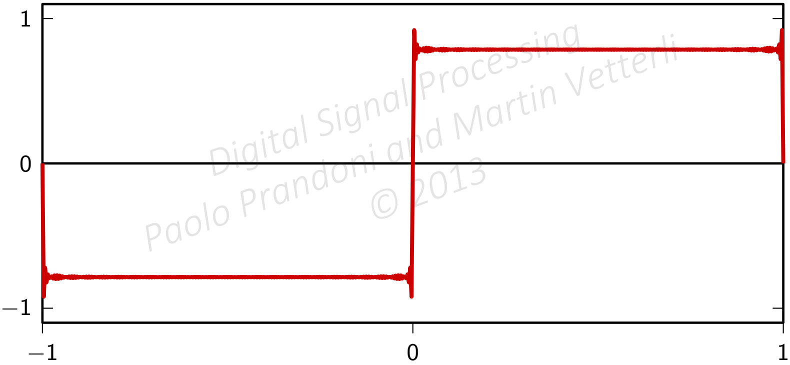
$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$



$N = 50$

Interesting questions when the space has ∞ dimensions

$$\sum_{k=0}^{N} \mathbf{x}^{(2k+1)}, \qquad \mathbf{x}^{(n)} = \sin(\pi\, n\, t)/n, \quad t \in [-1, 1]$$



$N = 150$

END OF MODULE 3.1

Digital Signal Processing

Module 3.2: Hilbert Space, properties and bases

Overview:

- Definition of Hilbert space
- Examples

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Hilbert Space – the ingredients:

1. a vector space: $H(V, \mathbb{C})$
2. an inner product: $\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{C}$
3. completeness

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We need *at least* to:

- resize vectors: scalar multiplication
- combine vectors together: addition

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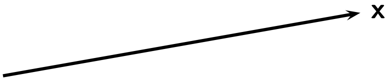
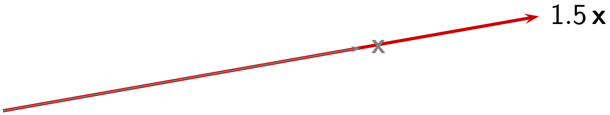
Scalar multiplication in $\mathbb{R}^2$

$$\mathbf{x} = [x_0 \quad x_1]^T$$



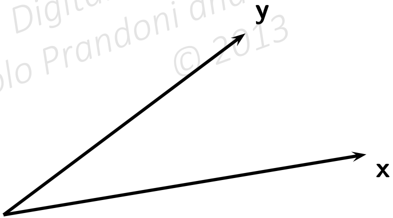
Scalar multiplication in $\mathbb{R}^2$

$$a\mathbf{x} = [ax_0 \quad ax_1]^T$$



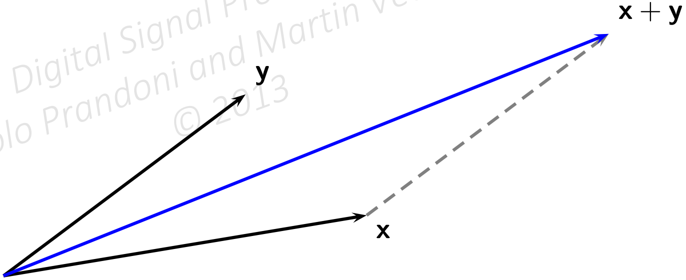
Addition in $\mathbb{R}^2$

$\mathbf{x}, \mathbf{y}$

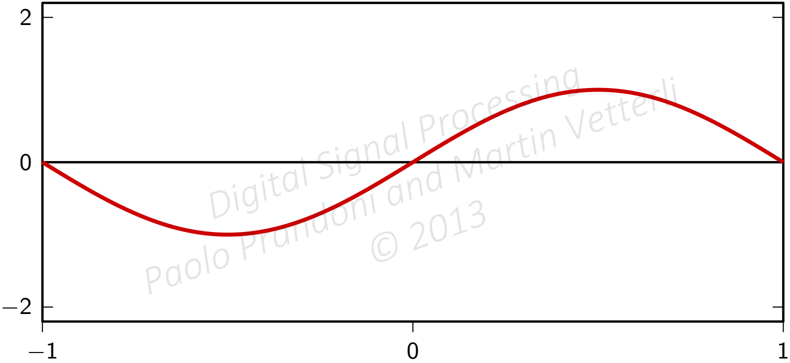


Addition in $\mathbb{R}^2$

$$\mathbf{x} + \mathbf{y} = [x_0 + y_0 \quad x_1 + y_1]^T$$



Scalar multiplication in $L_2[-1, 1]$



$$\mathbf{x} = \sin(\pi t)$$

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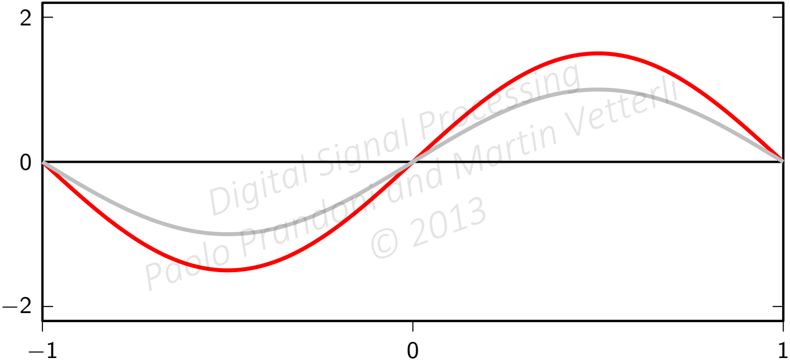
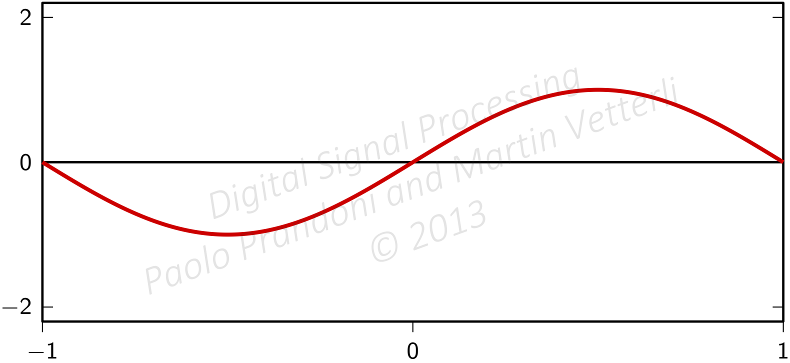


$1.5\,\mathbf{x}$

Addition in $L_2[-1, 1]$



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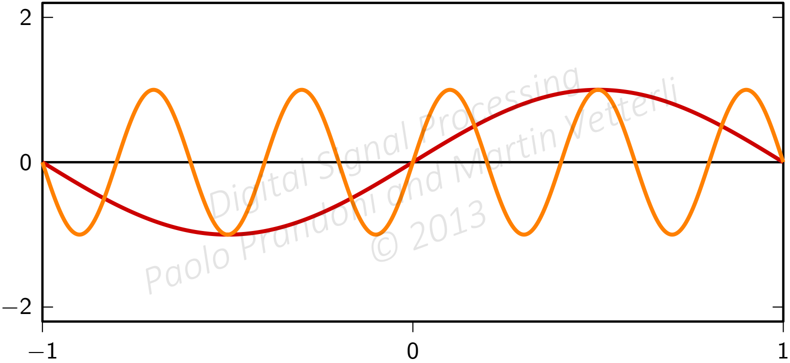

$$\mathbf{y} = \sin(5\pi t)$$

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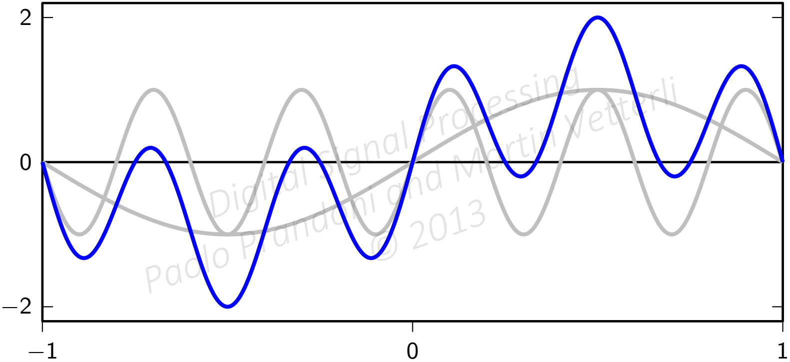

$$\mathbf{x}+\mathbf{y}$$

Formal properties of a vector space:

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{C}$:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
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- $\exists 0 \in V \quad | \quad \mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$
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Vector subspace:

- intuition: $\mathbb{R}^2 \subset \mathbb{R}^3$
- addition and scaling in subspace remain in subspace

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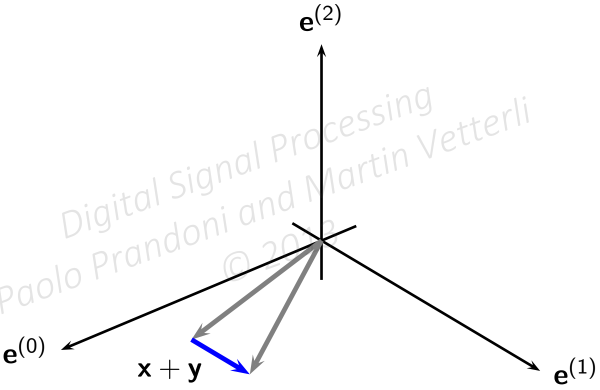
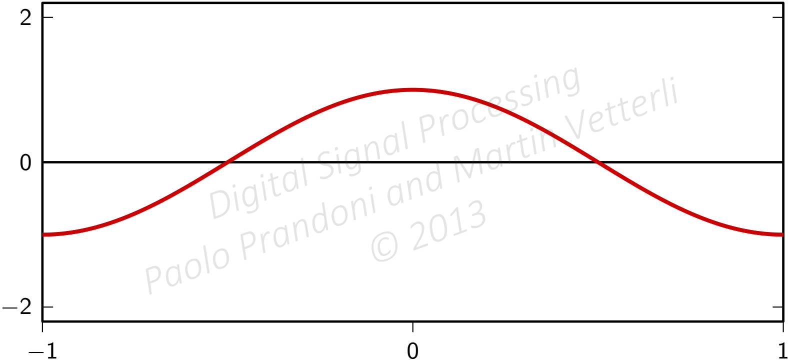
Addition in subspace:



Subspace of symmetric functions over $L_2[-1, 1]$



$$\mathbf{x} = \cos(\pi t)$$

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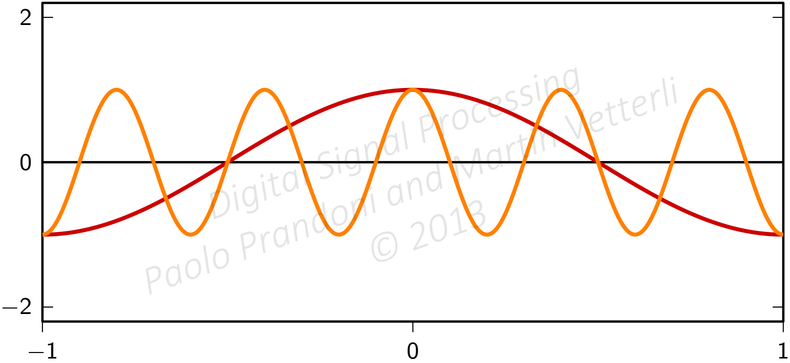
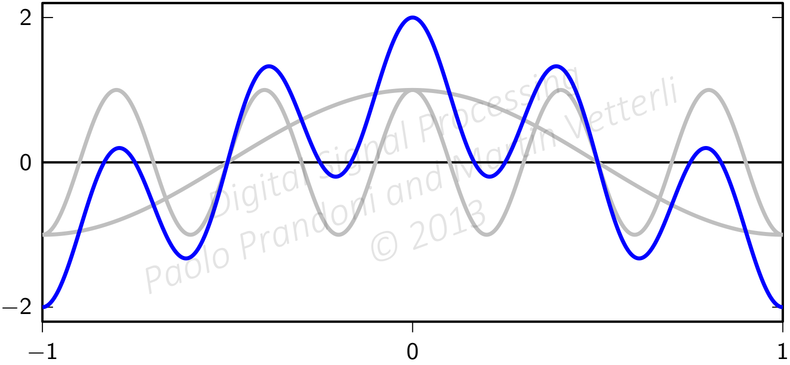


$\mathbf{y} = \cos(5\pi t)$

Subspace of symmetric functions over $L_2[-1, 1]$



$\mathbf{x} + \mathbf{y}$, symmetric

- measure of similarity between vectors
- when inner product is zero vectors are most different: orthogonal vectors

2) Inner product

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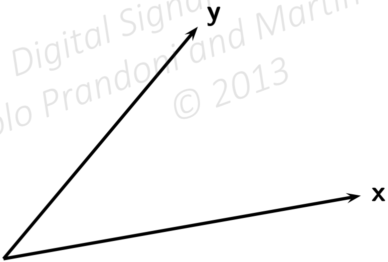
Inner product in $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0 y_0 + x_1 y_1$$



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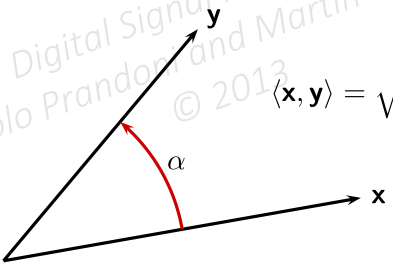

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sqrt{(x_0^2 + x_1^2)(y_0^2 + y_1^2)} \cos \alpha$$

Inner product in $\mathbb{R}^2$

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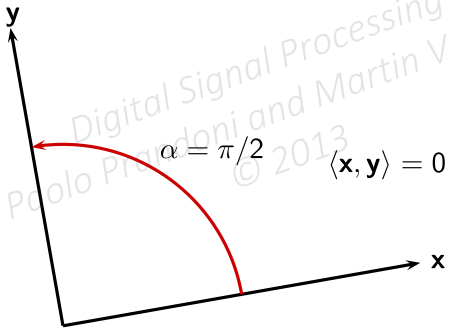
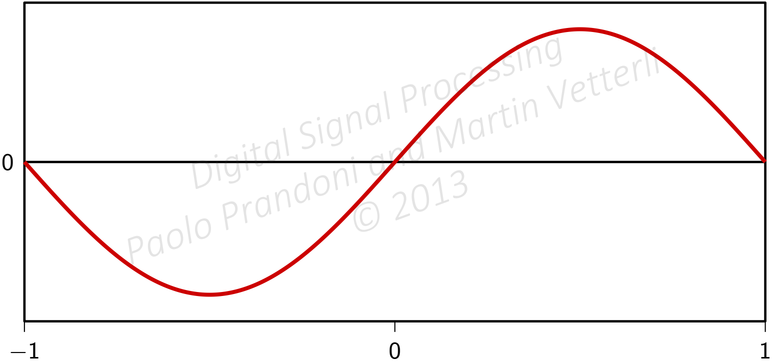


Inner product in $L_2[-1,1]$

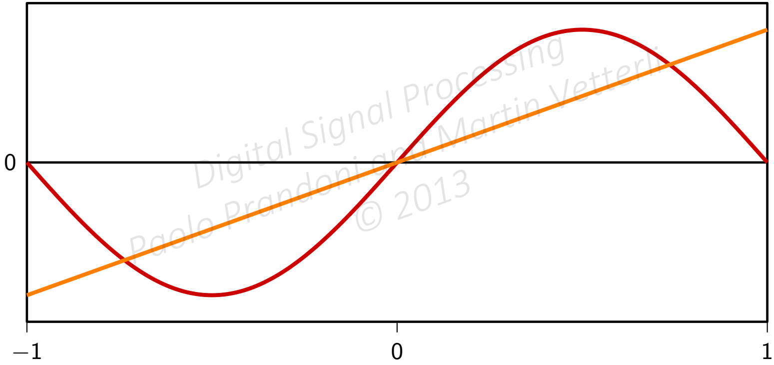
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t)dt$$



$$\mathbf{x} = \sin(\pi t)$$

Inner product in $L_2[-1,1]$

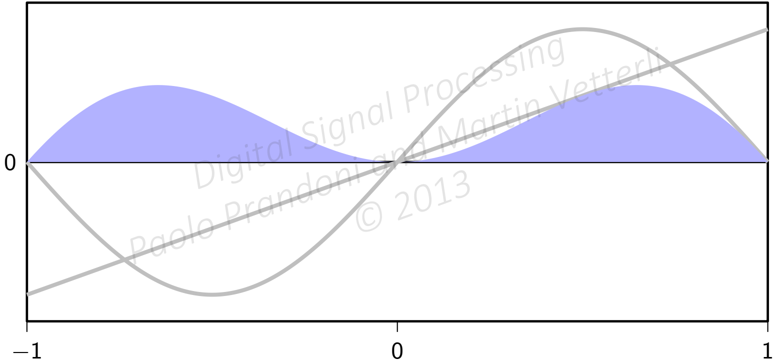
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} x(t)y(t)dt$$



$$\mathbf{y} = t$$

Inner product in $L_2[-1, 1]$

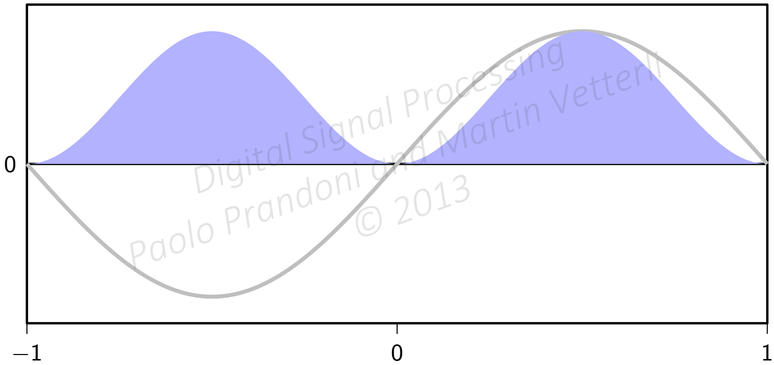
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-1}^{1} t \sin(\pi t) dt$$



$$\langle \mathbf{x}, \mathbf{y} \rangle = 2/\pi \approx 0.6367$$

Inner product in $L_2[-1, 1]$

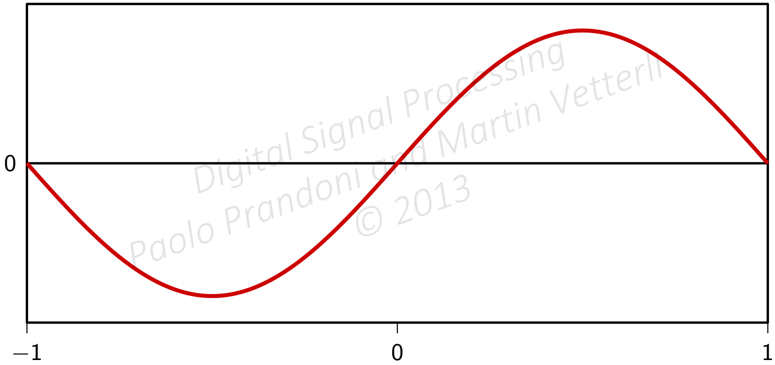
$\langle \mathbf{x}, \mathbf{x} \rangle$



$\mathbf{x} = \sin(\pi t), \ \langle \mathbf{x}, \mathbf{x} \rangle = 1$

Inner product in $L_2[-1, 1]$

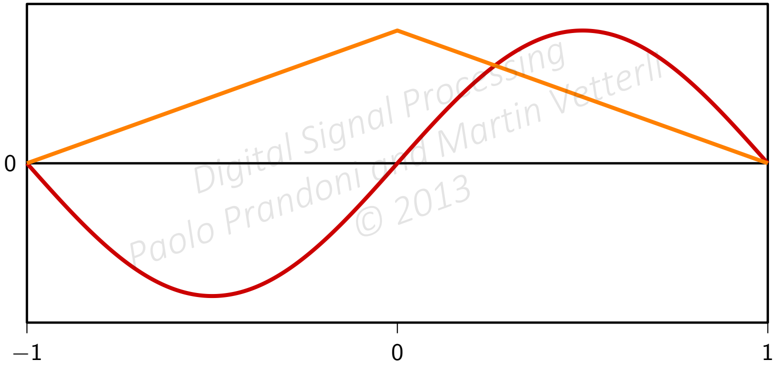
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces



$\mathbf{x} = \sin(\pi t)$, antisymmetric

Inner product in $L_2[-1, 1]$

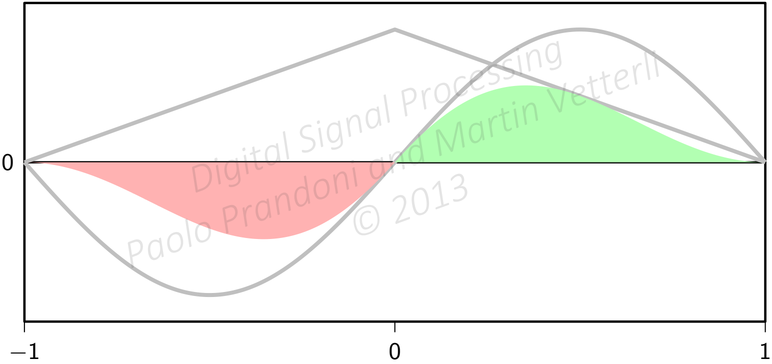
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces



$\mathbf{y} = 1 - |t|$, symmetric

Inner product in $L_2[-1, 1]$

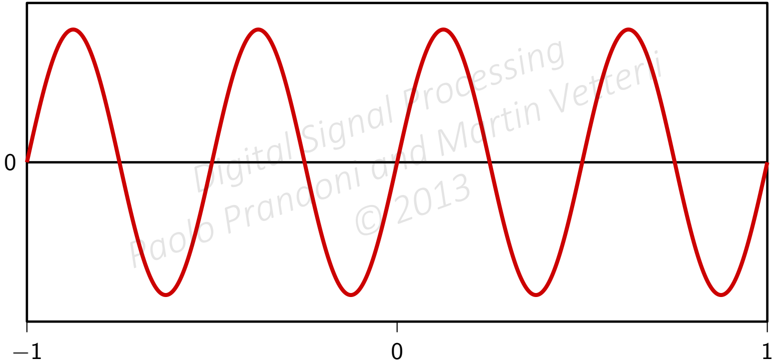
$\mathbf{x}, \mathbf{y}$ from orthogonal subspaces



$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

Inner product in $L_2[-1,1]$

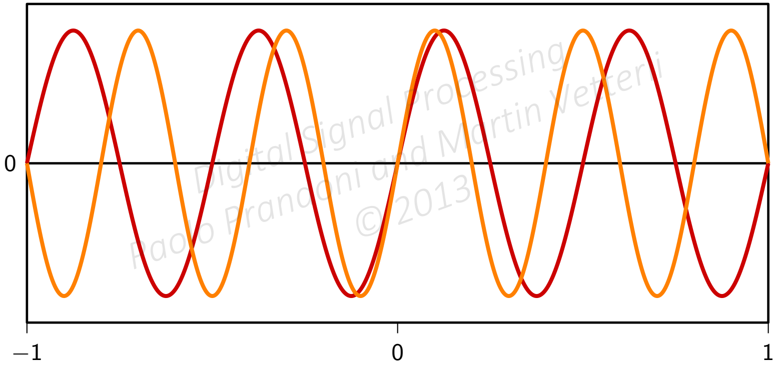
sinusoids with frequencies integer multiples of a fundamental



$\mathbf{x} = \sin(4\pi t)$

Inner product in $L_2[-1, 1]$

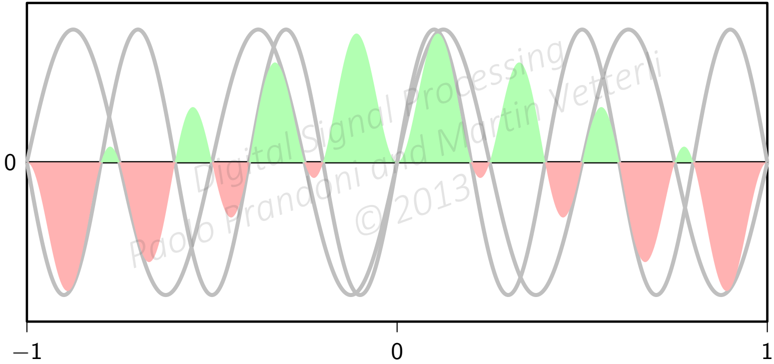
sinusoids with frequencies integer multiples of a fundamental



$\mathbf{x} = \sin(4\pi t)$, $\mathbf{y} = \sin(5\pi t)$

Inner product in $L_2[-1, 1]$

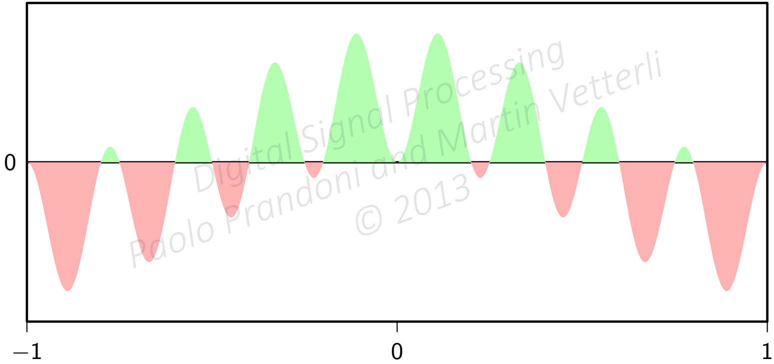
sinusoids with frequencies integer multiples of a fundamental



$\mathbf{x} = \sin(4\pi t)$, $\mathbf{y} = \sin(5\pi t)$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Inner product in $L_2[-1, 1]$

sinusoids with frequencies integer multiples of a fundamental



$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

Formal properties of the inner product

For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha \in \mathbb{C}$:

- $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$
- $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha^* \langle \mathbf{x}, \mathbf{y} \rangle$
  $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
- $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
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Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=0}^{N-1} x^*[n]y[n]$$

well defined for all finite-length vectors (i.e. vectors in $\mathbb{C}^N$)

Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

careful: sum may explode!

Inner product for signals

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=-\infty}^{\infty} x^*[n]y[n]$$

We require sequences to be *square-summable*: $\sum |x[n]|^2 < \infty$

Space of square-summable sequences: $\ell_2(\mathbb{Z})$

Norm

- inner product defines a norm: $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$
- norm defines a distance: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

Norm

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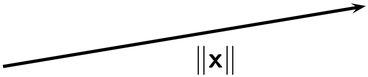
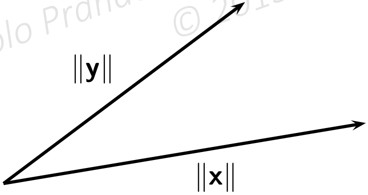
Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_0^2 + x_1^2}$$



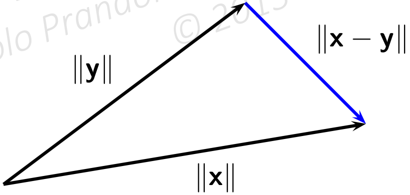
Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{y}\| = \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle} = \sqrt{y_0^2 + y_1^2}$$



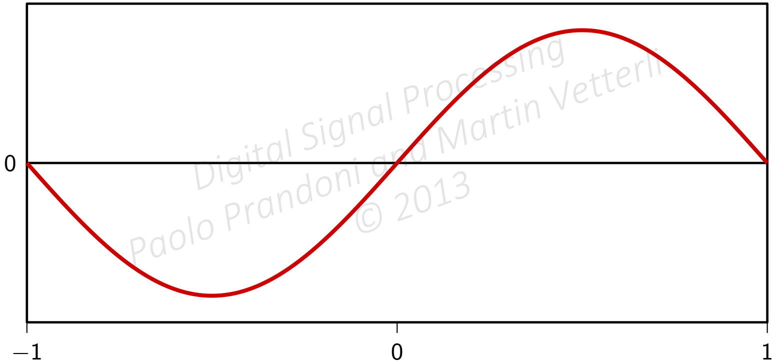
Norm and distance in $\mathbb{R}^2$

$$\|\mathbf{x}-\mathbf{y}\| = \sqrt{(x_0-y_0)^2+(x_1-y_1)^2}$$



Norm and distance in $L_2[-1,1]$

$$\|\mathbf{x}-\mathbf{y}\|^2 = \int_{-1}^{1} |x(t)-y(t)|^2 dt \text{ (MSE)}$$



$$\mathbf{x} = \sin(\pi t)$$

Norm and distance in $L_2[-1,1]$

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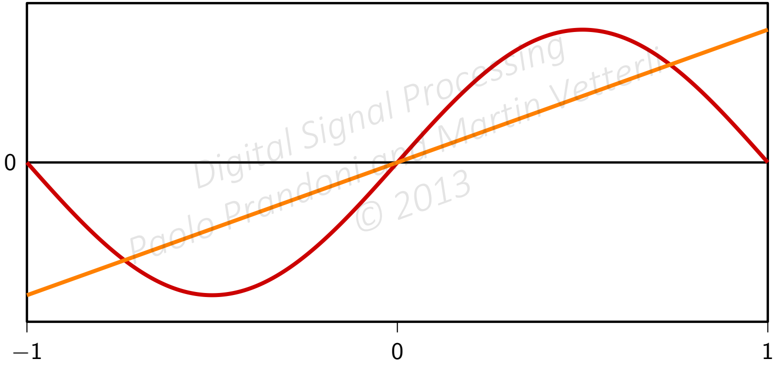

$$\mathbf{y} = t$$

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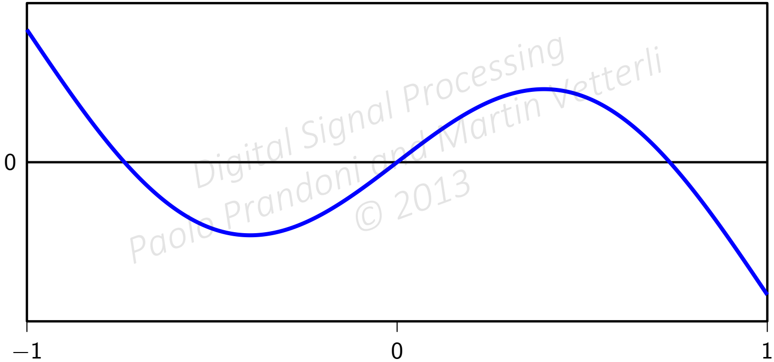

$\mathbf{x}-\mathbf{y}$

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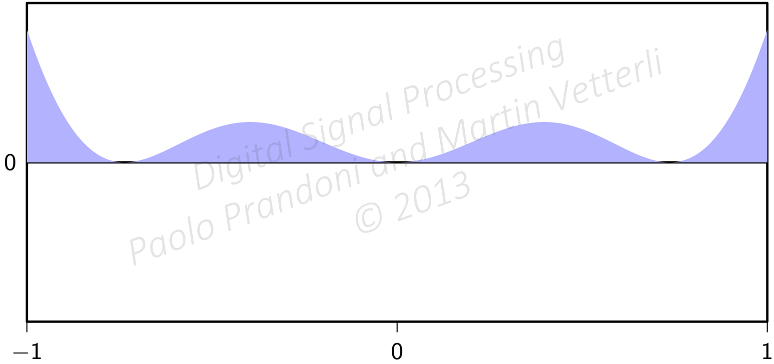
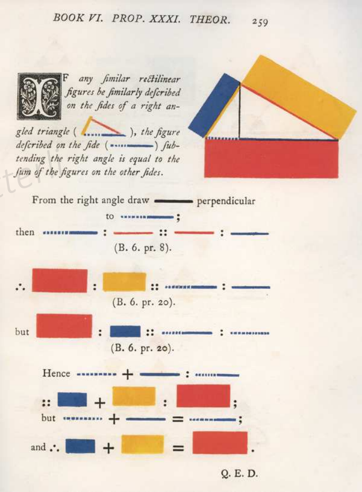


$$\|\mathbf{x}-\mathbf{y}\| = \sqrt{5/3 - 4/\pi} \approx 0.6272$$

A familiar result

Pythagorean theorem:

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \text{ for } \mathbf{x} \perp \mathbf{y}$$



From Euclid's elements by Oliver Byrne (1810 - 1880)

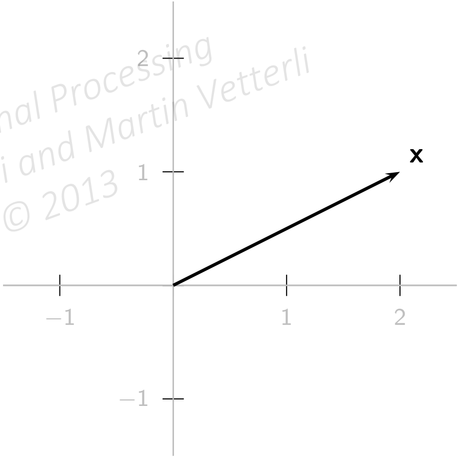
Bases

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{x} = 2\mathbf{e}^{(0)} + \mathbf{e}^{(1)}$$

$$\mathbf{x} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}$$

$$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)} \quad \text{for any } \alpha_0, \alpha_1$$



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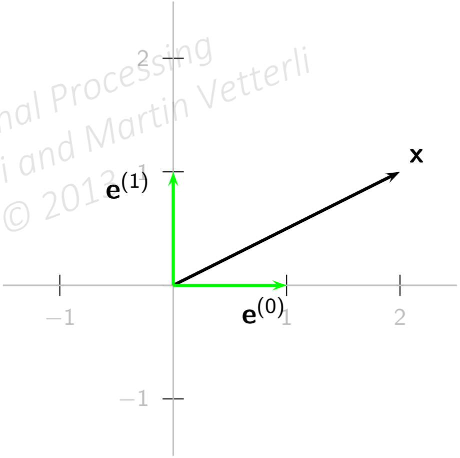

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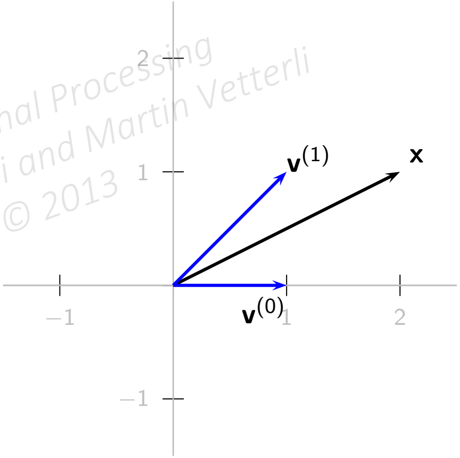

$\mathbf{x} \neq \alpha_0 \mathbf{g}^{(0)} + \alpha_1 \mathbf{g}^{(1)}$ for any α_0, α_1

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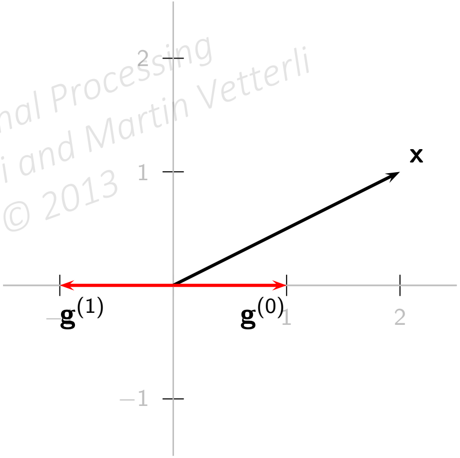

- vector space H
- set of K vectors from H: $W = \{\mathbf{w}^{(k)}\}_{k=0,1,\ldots,K-1}$

W is a basis for H if:

- we can write for *all* $\mathbf{x} \in H$:

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}, \quad \alpha_k \in \mathbb{C}$$

- the coefficients α_k are unique

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Bases

Unique representation implies linear independence:

$$\sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)} = 0 \quad \Longleftrightarrow \quad \alpha_k = 0, \;\; k = 0, 1, \ldots, K-1$$

Orthogonal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = 0 \text{ for } k \neq n$$

Orthonormal basis:

$$\langle \mathbf{w}^{(k)}, \mathbf{w}^{(n)} \rangle = \delta[n-k]$$

We can always orthonormalize a basis via the Gram-Schmidt algorithm.

Special bases

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Basis expansion

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

how do we find the α's ?

Orthonormal bases are the best:

$$\alpha_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle$$

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Example: bases for $\mathbb{C}^N$

- a basis will contain N vectors
- canonical (orthonormal) basis:

$$\mathbf{e}^{(k)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

k-th position, $0 \leq k < N$

Example: bases for sequences in $\ell_2(\mathbb{Z})$

- a basis will contain infinite vectors
- canonical (orthonormal) basis:

$$\mathbf{e}^{(k)} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

k-th position, $k \in \mathbb{Z}$

Completeness

limiting operations must yield vector space elements

Example of an *incomplete* space: the set of rational numbers

$$x_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q} \quad \text{but} \quad \lim_{n\to\infty} x_n = e \notin \mathbb{Q}$$

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END OF MODULE 3.2

Digital Signal Processing

Module 3.3: Hilbert Space and approximation

Overview:

- Norm conservation, Parseval
- Approximation by projection
- Examples

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Parseval's Theorem

$$\mathbf{x} = \sum_{k=0}^{K-1} \alpha_k \mathbf{w}^{(k)}$$

For an orthonormal basis:

$$\|\mathbf{x}\|^2 = \sum_{k=0}^{K-1} |\alpha_k|^2$$

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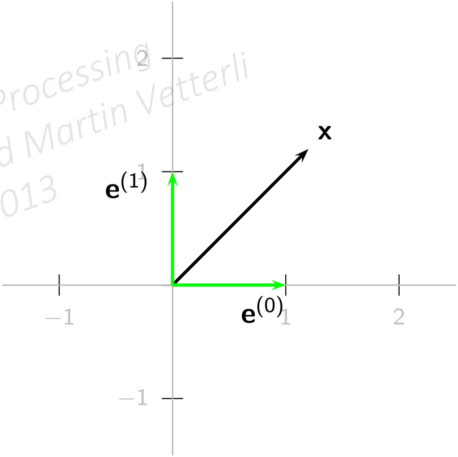
Example

- canonical basis $E = \{\mathbf{e}^{(0)}, \mathbf{e}^{(1)}\}$
- $\mathbf{x} = \alpha_0 \mathbf{e}^{(0)} + \alpha_1 \mathbf{e}^{(1)}$
- new basis $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$ with

$$\mathbf{v}^{(0)} = [\cos\theta \;\; \sin\theta]^T$$

$$\mathbf{v}^{(1)} = [-\sin\theta \;\; \cos\theta]^T$$

- $\mathbf{x} = \beta_0 \mathbf{v}^{(0)} + \beta_1 \mathbf{v}^{(1)}$



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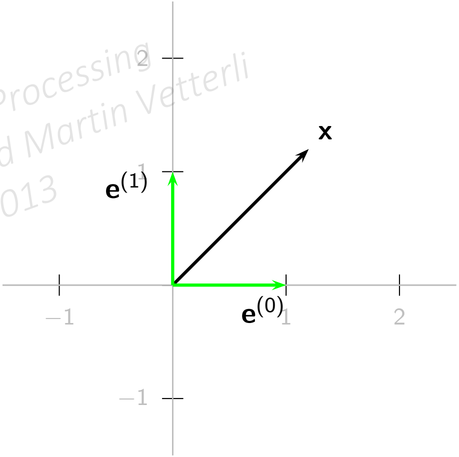

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- new basis $V = \{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}\}$ with

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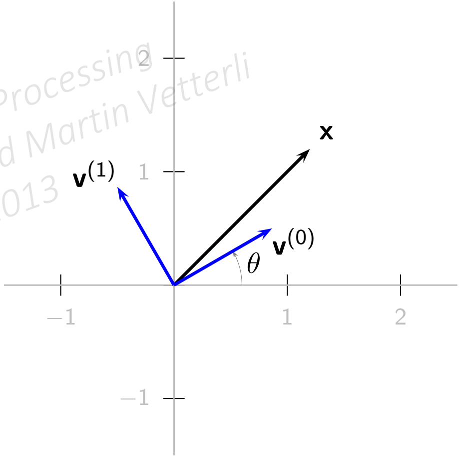

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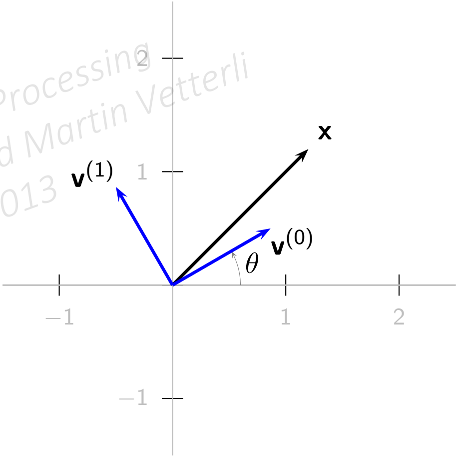


Example

- new basis is orthonormal:

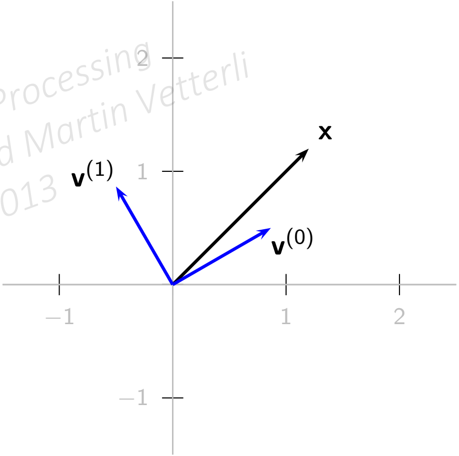
$$\beta_0 = \langle \mathbf{v}^{(0)}, \mathbf{x} \rangle$$

$$\beta_1 = \langle \mathbf{v}^{(1)}, \mathbf{x} \rangle$$

- in compact form:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \mathbf{R}\alpha$$

- $\mathbf{R}$: rotation matrix
- key fact: $\mathbf{R}^T\mathbf{R} = \mathbf{I}$



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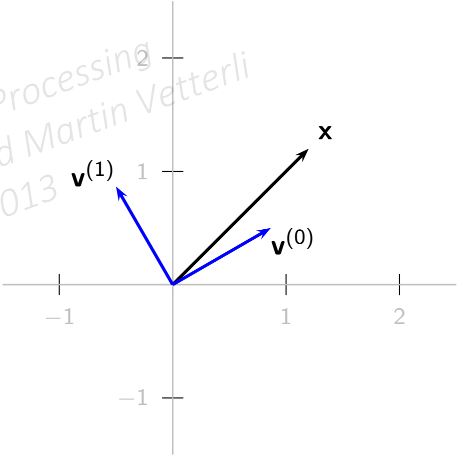

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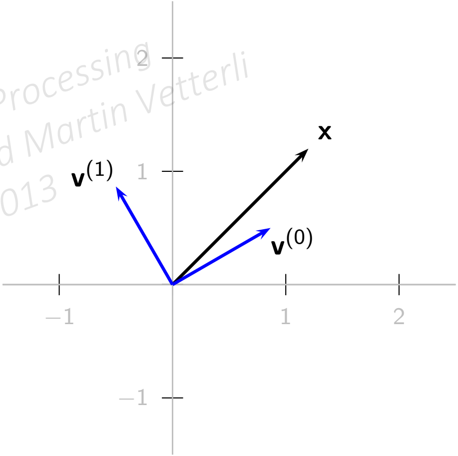

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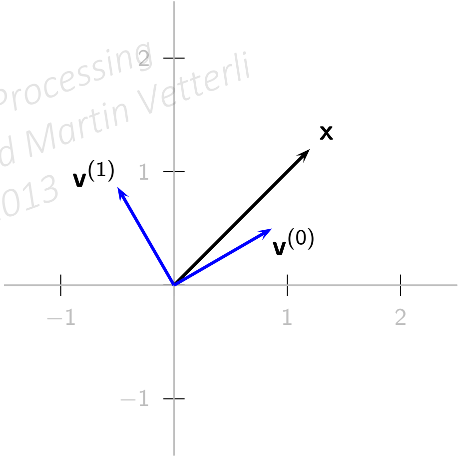

Example

- square norm in canonical basis: $\|\mathbf{x}\|^2 = \alpha_0^2 + \alpha_1^2$
- square norm in rotated basis: $\|\mathbf{x}\|^2 = \beta_0^2 + \beta_1^2$
- let's verify Parseval:

$$
\begin{aligned}
\beta_0^2 + \beta_1^2 &= \boldsymbol{\beta}^T \boldsymbol{\beta} \\
&= (\mathbf{R}\boldsymbol{\alpha})^T (\mathbf{R}\boldsymbol{\alpha}) \\
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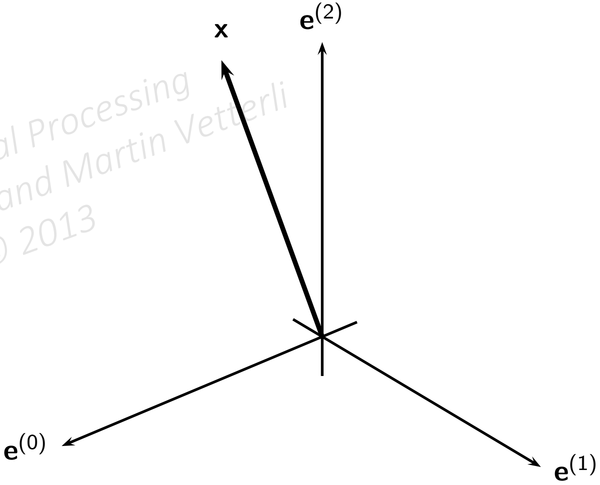
- subspace $S \subseteq V$
- approximate $\mathbf{x}$ with $\hat{\mathbf{x}} \in S$



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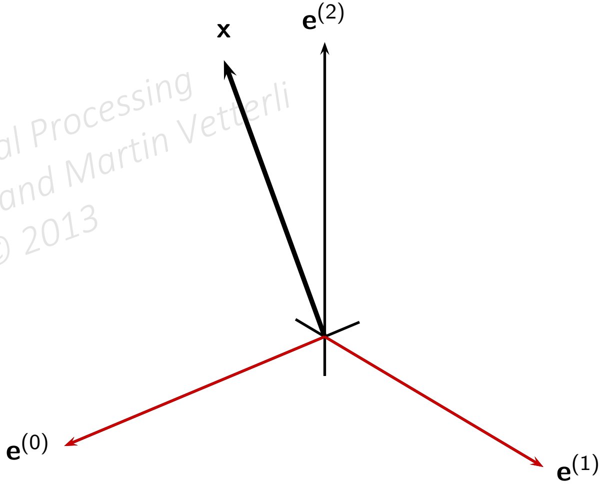

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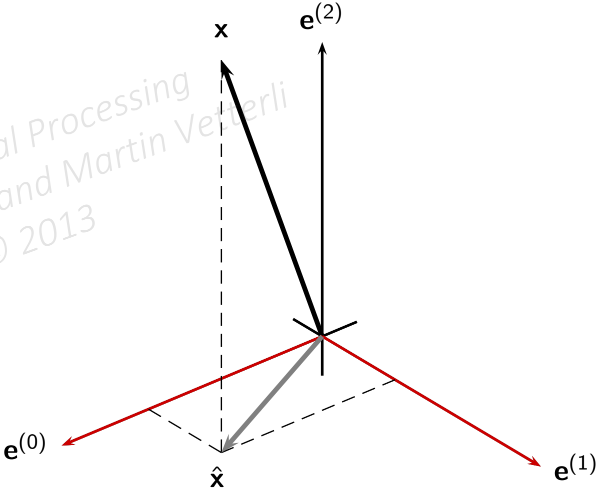

- $\{\mathbf{s}^{(k)}\}_{k=0,1,\ldots,K-1}$ orthonormal basis for S
- orthogonal projection:

$$\hat{\mathbf{x}} = \sum_{k=0}^{K-1} \langle \mathbf{s}^{(k)}, \mathbf{x} \rangle \, \mathbf{s}^{(k)}$$

orthogonal projection is the "best" approximation over S

Least-Squares Approximation

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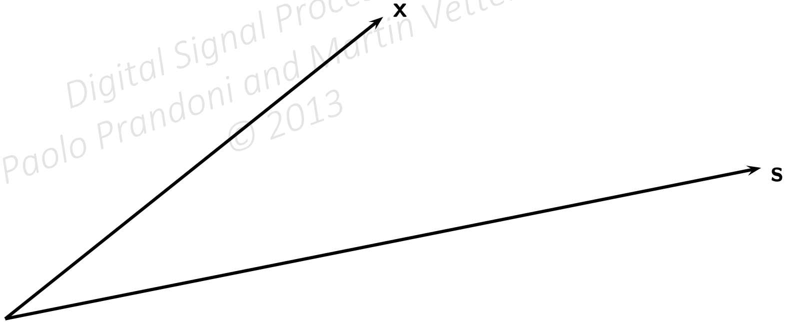

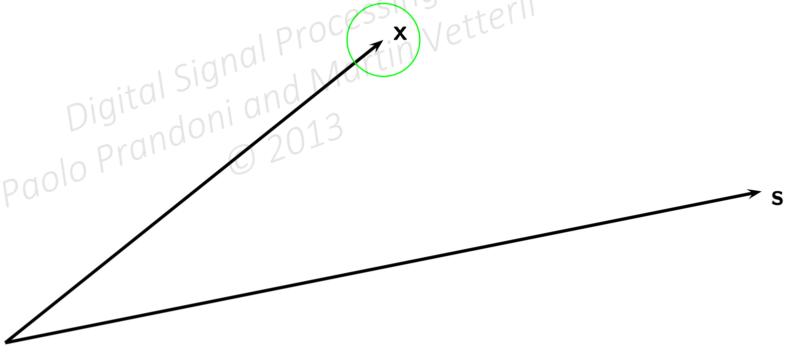
x
s

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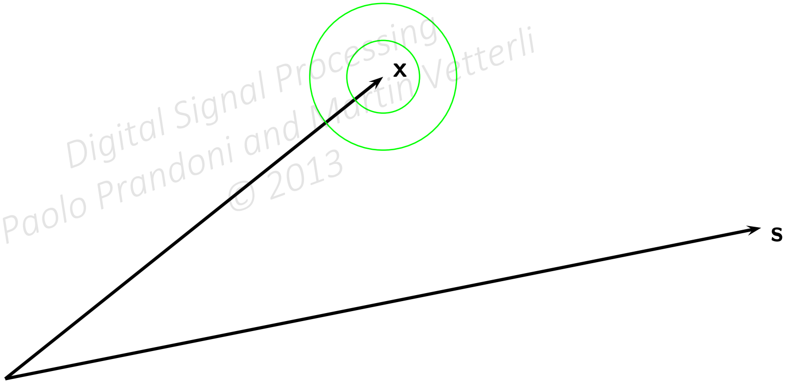

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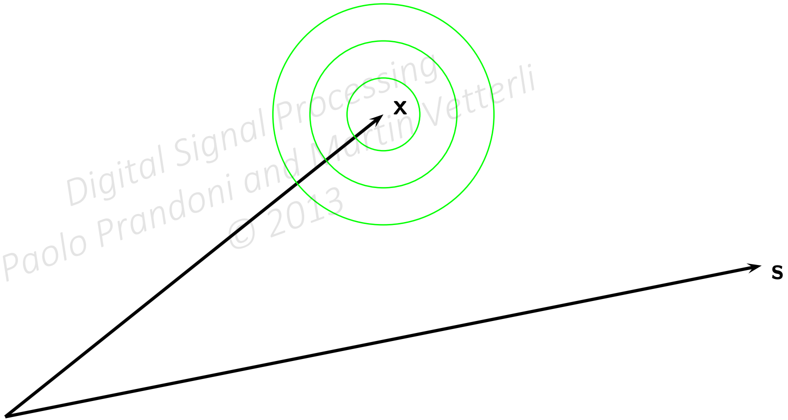

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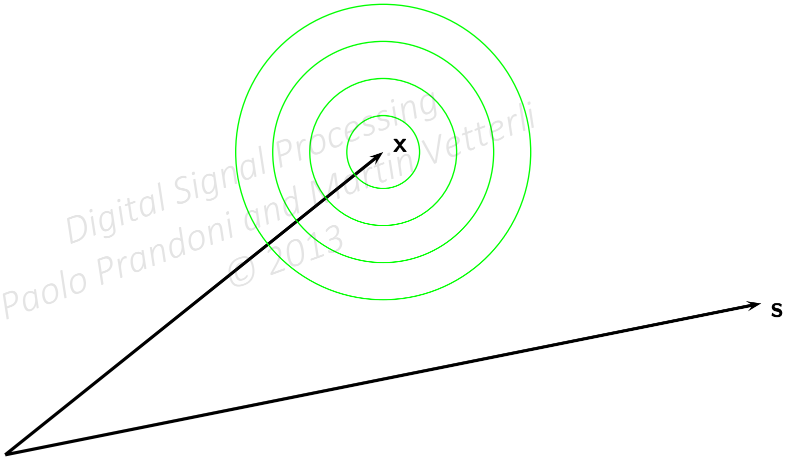

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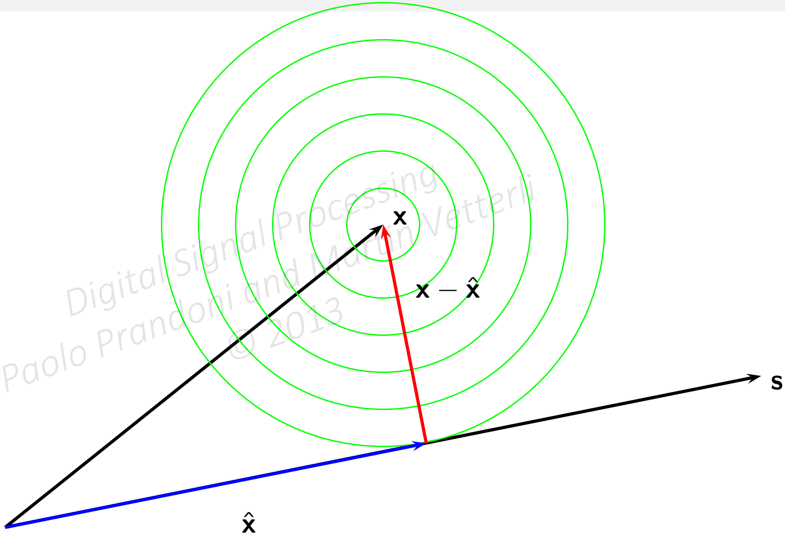

Example: polynomial approximation

- Hilbert space $P_N[-1,1] \subset L_2[-1,1]$
- a self-evident, naive basis: $\mathbf{s}^{(k)} = t^k, \quad k = 0, 1, \ldots, N-1$
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The Gram-Schmidt algorithm leads to an orthonormal basis for $P_N([-1,1])$
(see appendix if interested in details)

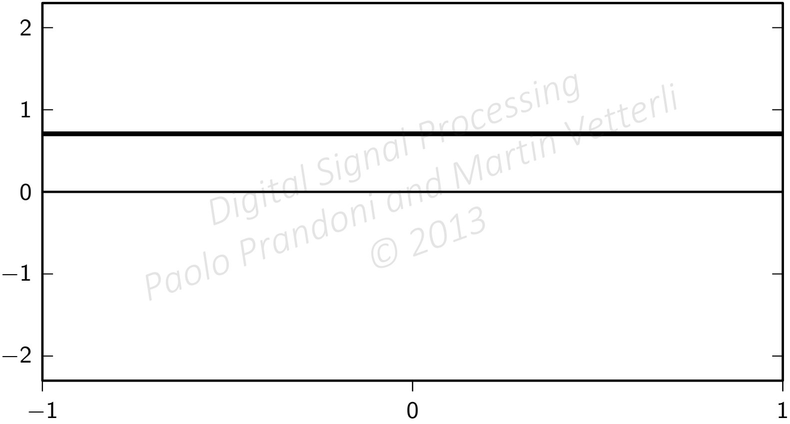
$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\,t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \ldots$$

Legendre Polynomials



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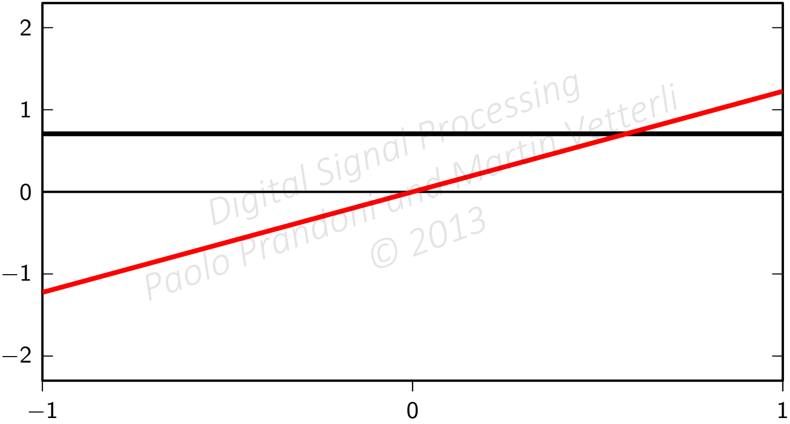

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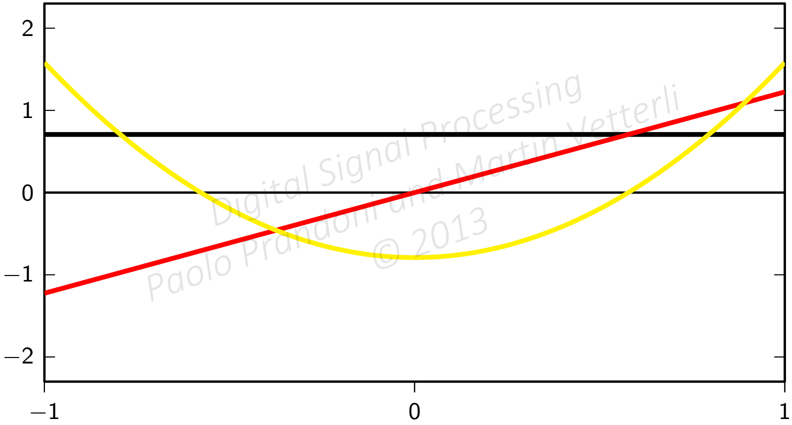

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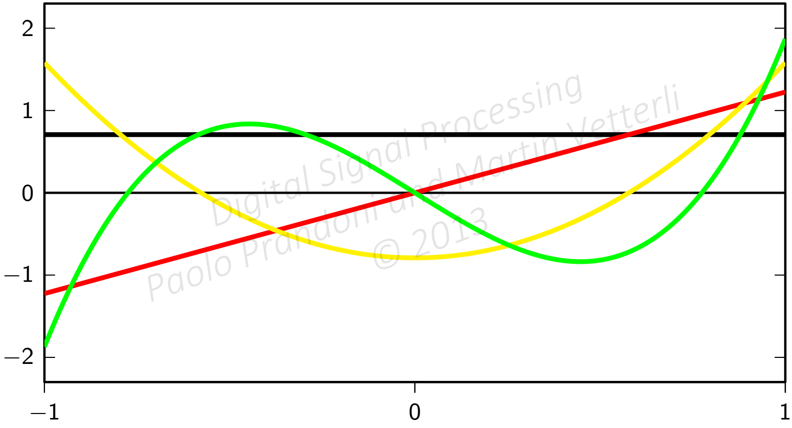

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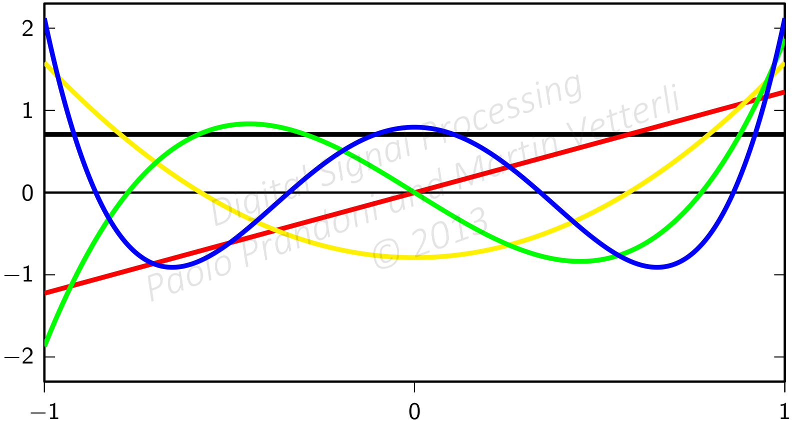

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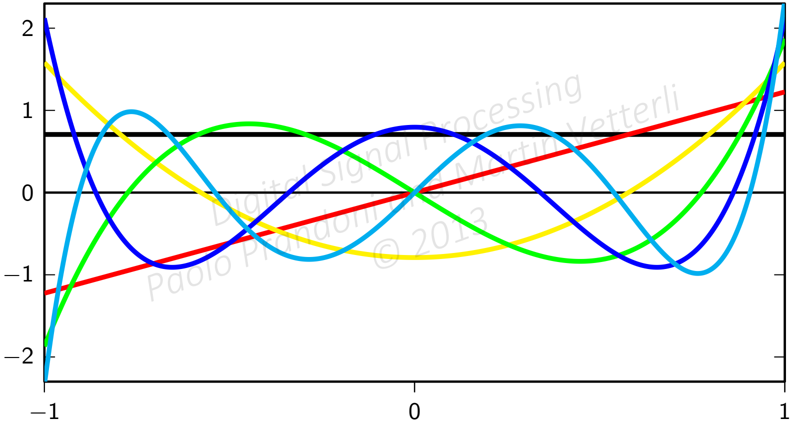

Orthogonal projection over $P_3[-1, 1]$

$$\alpha_k = \langle \mathbf{u}^{(k)}, \mathbf{x} \rangle = \int_{-1}^{1} u_k(t) \sin t \, dt$$

- $\alpha_0 = \langle \sqrt{1/2}, \sin t \rangle = 0$
- $\alpha_1 = \langle \sqrt{3/2}\, t, \sin t \rangle \approx 0.7377$
- $\alpha_2 = \langle \sqrt{5/8}(3t^2 - 1), \sin t \rangle = 0$

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Approximation

Using the orthogonal projection over $P_3[-1,1]$:

$$\sin t \rightarrow \alpha_1 \mathbf{u}^{(1)} \approx 0.9035\, t$$

Using Taylor's series:

$$\sin t \approx t$$

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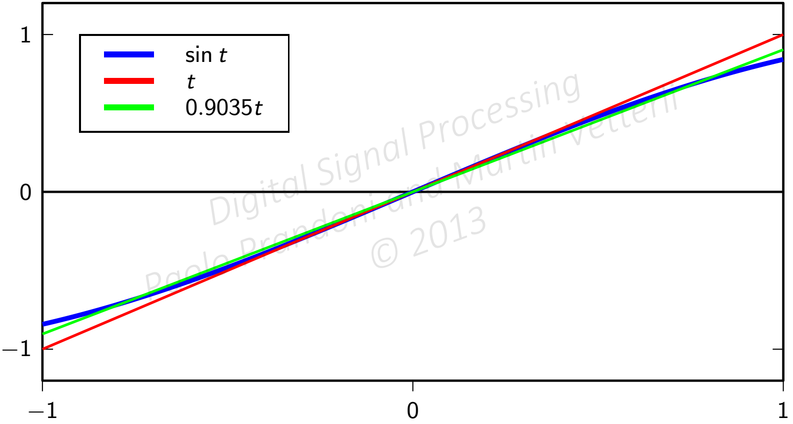
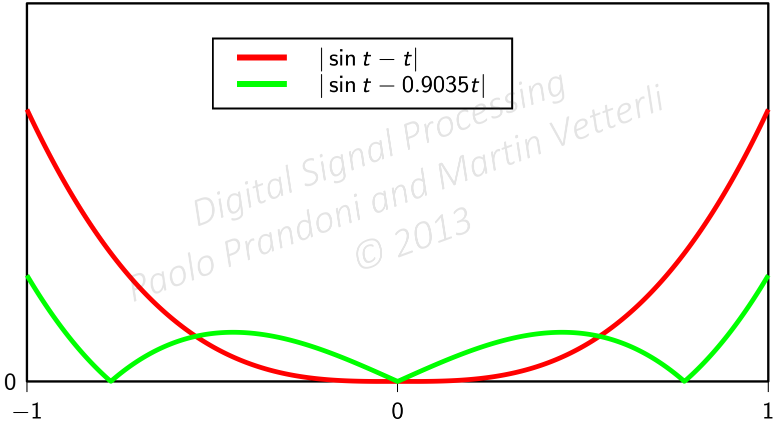
Sine approximation



Approximation error



Error norm

Orthogonal projection over $P_3[-1,1]$:

$$\|\sin t - \alpha_1 \mathbf{u}^{(1)}\| \approx 0.0337$$

Taylor series:

$$\|\sin t - t\| \approx 0.0857$$

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- finite-length and periodic signals live in $\mathbb{C}^N$
- infinite-length signals live in $\ell_2(\mathbb{Z})$
- different bases are different observation tools for signals
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END OF MODULE 3.3

Appendix: orthonormalization of the naive polynomial basis

Building an orthonormal basis

Gram-Schmidt orthonormalization procedure:

$$\{\mathbf{s}^{(k)}\} \longrightarrow \{\mathbf{u}^{(k)}\}$$

original set orthonormal set

Algorithmic procedure: at each step k

1. $\mathbf{p}^{(k)} = \mathbf{s}^{(k)} - \sum_{n=0}^{k-1} \langle \mathbf{u}^{(n)}, \mathbf{s}^{(k)} \rangle \, \mathbf{u}^{(n)}$

2. $\mathbf{u}^{(k)} = \mathbf{p}^{(k)} / \|\mathbf{p}^{(k)}\|$

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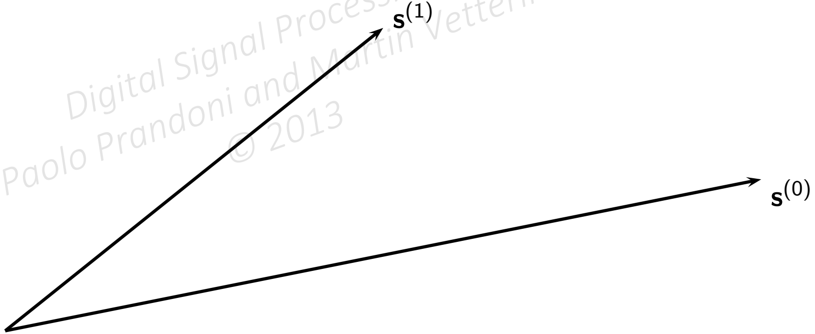

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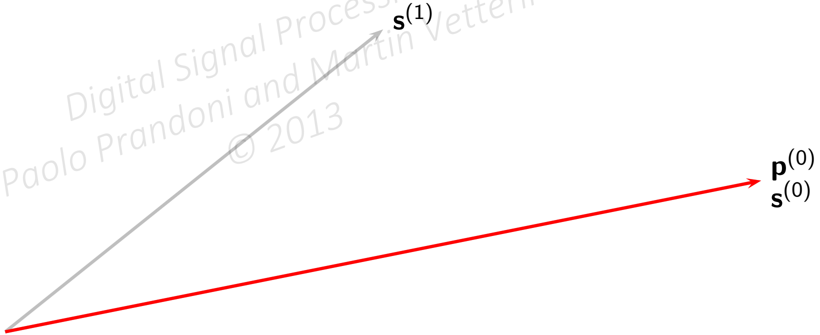

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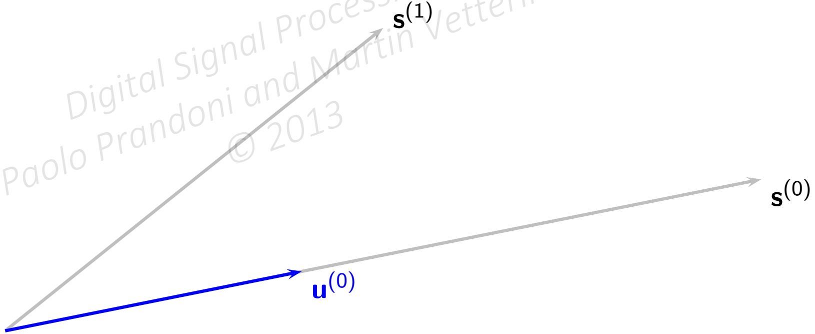

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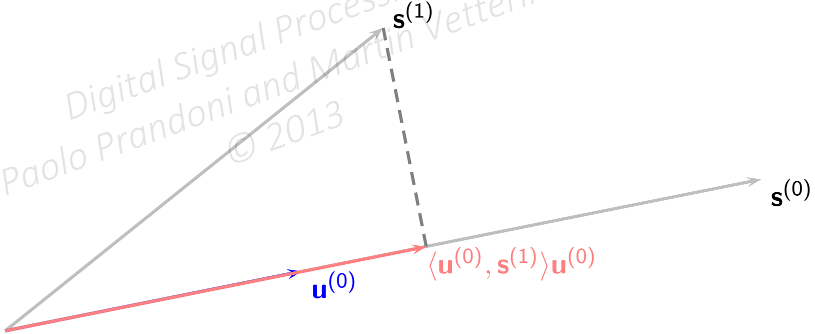

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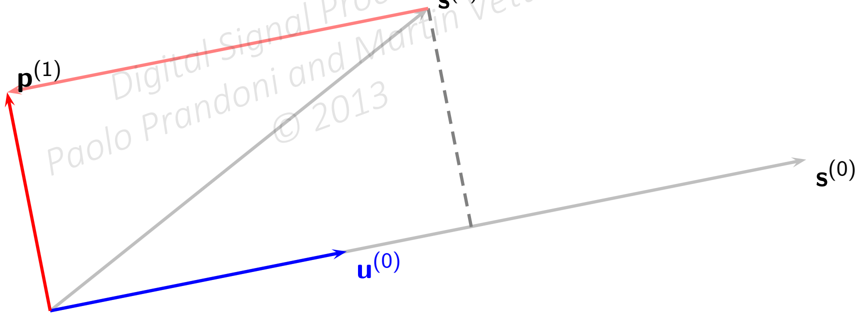

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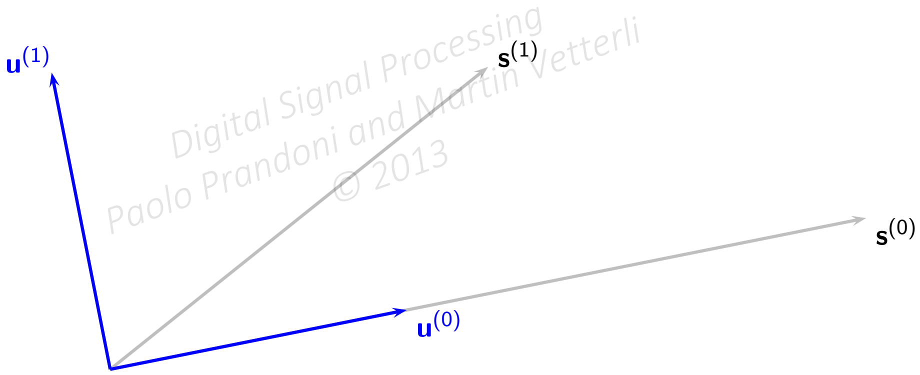

Building an orthonormal basis

Gram-Schmidt orthonormalization of the naive basis: $\{\mathbf{s}^{(k)}\} \rightarrow \{\mathbf{u}^{(k)}\}$

- $\mathbf{s}^{(0)} = 1$
  - $\mathbf{p}^{(0)} = \mathbf{s}^{(0)} = 1$
  - $\|\mathbf{p}^{(0)}\|^2 = 2$
  - $\mathbf{u}^{(0)} = \mathbf{p}^{(0)}/\|\mathbf{p}^{(0)}\| = \sqrt{1/2}$
- $\mathbf{s}^{(1)} = t$
  - $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(1)} \rangle = \int_{-1}^{1} t/\sqrt{2} = 0$
  - $\mathbf{p}^{(1)} = \mathbf{s}^{(1)} = t$
  - $\|\mathbf{p}^{(1)}\|^2 = 2/3$
  - $\mathbf{u}^{(1)} = \sqrt{3/2}\, t$
- $\mathbf{s}^{(2)} = t^2$
  - $\langle \mathbf{u}^{(0)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^{1} t^2/\sqrt{2} = 2/3\sqrt{2}$
  - $\langle \mathbf{u}^{(1)}, \mathbf{s}^{(2)} \rangle = \int_{-1}^{1} t^3/\sqrt{2} = 0$
  - $\mathbf{p}^{(2)} = \mathbf{s}^{(2)} - (2/3\sqrt{2})\mathbf{u}^{(0)} = t^2 - 1/3$
  - $\|\mathbf{p}^{(2)}\|^2 = 8/45$
  - $\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$

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Legendre polynomials

$$\mathbf{u}^{(0)} = \sqrt{1/2}$$

$$\mathbf{u}^{(1)} = \sqrt{3/2}\,t$$

$$\mathbf{u}^{(2)} = \sqrt{5/8}(3t^2 - 1)$$

$$\mathbf{u}^{(3)} = \ldots$$

END OF MODULE 3